Acceptable Complexity Measures of Theorems

Bruno Grenet

Bruno.Grenet@ens-lyon.fr
http://perso.ens-lyon.fr/bruno.grenet/

École Normale Supérieure de Lyon, France

Bloomington – October 31, 2008

Historical Overview

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- 2005: Calude and Jürgensen prove the “heuristic principle”

Goal

- $\delta(x) = H(x) - |x|$ where H is the *program-size* complexity.

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- Is it the only measure satisfying the heuristic principle?

Outline

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- Gödel numbering for the language L: computable one-to-one function $g : L \to X_2^*$
- G: set of all the Gödel numberings

Self-delimiting Turing Machines

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Self-delimiting Turing Machine: $PROG_T$ is prefix-free

Program-size complexity

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$$H_i \triangleq H_{i,U_i}$$

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Definition

$$\delta_g(u) = H_2(g(u)) - \lceil \log_2(i) \cdot |x|_i \rceil \,,$$

where g is a Gödel numbering.

Invariance of the measure

Theorem

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- For every g and g', there exists a constant d such that

$$\left|H_2(g(u)) - H_2(g'(u))\right| \leq d \text{ and } \left|\delta_g(u) - \delta_{g'}(u)\right| \leq d + 1.$$

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Proposition

$\forall N > 0, \lim_{n\to\infty} i^{-n} \cdot \mathrm{card}\,\{x \in X_i^* : |x|_i = n, \delta_g(x) \leq N\} = 0$

Outline

Complexity Measure Builder

Definition

Let $\hat{\rho}_i : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ be a computable function. Then we define the *complexity measure builder* ρ by

$$\begin{aligned} \rho : G &\to [X_i^* \to \mathbb{Q}] \\ g &\mapsto \rho_g \end{aligned}$$

where $\rho_g(u) = \hat{\rho}_i(H_2(g(u)), |u|_i)$.

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- $\hat{\rho}_i$: *witness* of the builder
- ρ_g: complexity measure

Acceptable Builder

(i) If $\mathcal{F} \vdash x$, then $\rho_{\mathbf{g}}(x) < N_{\mathcal{F}}$.

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The function δ_g is an acceptable complexity measure.

Proposition

The program-size complexity is not an acceptable complexity measure.

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Theorem

The three conditions are independent from each other.

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Introduction

Can we find other acceptable measures of complexity?

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Proposition

Suppose that ρ_g is acceptable. Then so is $\alpha \cdot \rho_g + \beta$, $\alpha, \beta \in \mathbb{Q}$, $\alpha > 0$.

Results

Proposition

Let $\hat{\rho}_i : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ be a computable function, linear in both variables. If it defines an acceptable complexity measure, then

$$\hat{\rho}_i(x, y) = a \cdot (x - \varepsilon \cdot \lceil \log_2(i) \cdot y \rceil) + b,$$

where $1/2 \leq \varepsilon \leq 1$.

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Proposition

Let $\rho_g(x) = H_2(g(x))/f(|x|_i)$ where f is computable. Then ρ_g is not acceptable.

Summary of the work

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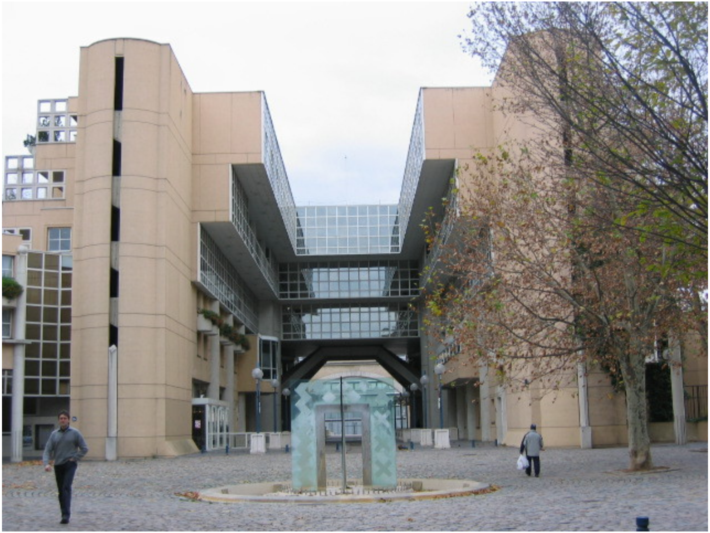
  - Some corrections
  - Key elements in the proofs
- Proposition of a general definition of *acceptable complexity measure of theorems*
- Studying those acceptable measures to find other ones (in progress)

Thank you for your attention!



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