1. (20 points) Assume you flip 1000 pennies and 1000 dimes. You get to keep the coins that land head-up.
   (a) (10 points) What's the average amount of cents you get to keep ?
   (b) (10 points) What's the variance of the amount of cents you get to keep ?

   Solution: Let X and Y be random variables representing the number of heads we get for the pennies and dimes, respectively. If Z is the random variable representing the amount of cents we get to keep, then $Z = X + 10Y$. Part a then asks for $\mathbb{E}[Z] = \mathbb{E}[X+10Y]$ and part b asks for $\text{Var}[Z] = \text{Var}[X+10Y]$. Now, our random variables X and Y are both random variables from the binomial distribution with $n = 1000$ and $p = \frac{1}{2}$. Therfore, $\mathbb{E}[X] = \mathbb{E}[Y] = np = 500$. And so,

$$\mathbb{E}[X + 10Y] = \mathbb{E}[X] + 10\mathbb{E}[Y] = 500 + 5000 = 5500 \text{ (cents)}$$

   Since X and Y are independent random variables, $\text{Var}[X + 10Y] = \text{Var}[X] + \text{Var}[10Y]$. Now X and Y are random variables from the binomial distribution, so $\text{Var}[X] = \text{Var}[Y] = np(1 - p) = 250$. We therefore have

$$\text{Var}[X + 10Y] = \text{Var}[X] + 100\text{Var}[Y] = 250 + 25000 = 25250 \text{ (cents}^2\text{)}.$$

2. (20 points) You have items of Type 1 and Type 2. Each item is of either Type 1 or Type 2, but you can't directly determine its type. Each item has three properties: A, B, and C. Each property has one of two values: '+' and '-'. The following table gives the probability that an item is of Type 1 depending on the value of a single property.

	'+'	'-'
A	0.9	0.1
B	0.8	0.2
C	0.7	0.3

   Assume that the properties are statistically independent.
   (a) (5 points) If an item is of Type 1, how likely is it that all of the following will be true: A is '+', B is '+', and C is '+' ?
   (b) (5 points) If an item is of Type 1, how likely is it that at least one of the following will be true: A is '+', B is '+', or C is '+' ?
   (c) (5 points) Suppose we say an item is in Class 1 when both A and B have the value '+'. What fraction of the items in Class 1 are of Type 1 ?
   (d) (5 points) Suppose we say an item is in Class 1 when at least two of A, B, and C have the value '+'. What fraction of the items in Class 1 are of Type 1 ?

   Solution This is the original version of the question. Let A_+ and A_- be the event that the item's property A is of value '+' and '-', respectively. We also define B_+, B_-, C_+, and C_- in an analogous manner. The table of values given above is then just the values of $\Pr(T_1|X)$ where T_1 is the event corresponding to an item being of Type 1, and $X \in \{A_+, A_-, B_+, B_-, C_+, C_-\}$. Part a then asks for $\Pr(A_+ \cap B_+ \cap C_+ \mid T_1)$ while Part b asks for $\Pr(A_+ \cup B_+ \cup C_+ \mid T_1)$. Now, from the formula for conditional probability, we have

$$\Pr(A_+ \cap B_+ \cap C_+ \mid T_1) = \frac{\Pr(A_+ \cap B_+ \cap C_+ \cap T_1)}{\Pr(T_1)} = \frac{\Pr(T_1 \mid A_+ \cap B_+ \cap C_+) \times \Pr(A_+ \cap B_+ \cap C_+)}{\Pr(T_1)}$$

   Since we are not given both $\Pr(A_+ \cap B_+ \cap C_+)$ and $\Pr(T_1)$ we can't find $\Pr(A_+ \cap B_+ \cap_{C+} \mid T_1)$ completely. However, $\Pr(T_1 \mid A_+ \cap B_+ \cap C_+) = \Pr(T_1|A_+) \times \Pr(T_1|B_+) \times \Pr(T_1|C_+)$ by independence and so we have

$$\begin{aligned}\Pr(A_+ \cap B_+ \cap C_+ \mid T_1) &= \frac{\Pr(T_1 \mid A_+ \cap B_+ \cap C_+) \times \Pr(A_+ \cap B_+ \cap C_+)}{\Pr(T_1)} \\ &= \frac{\Pr(T_1|A_+) \times \Pr(T_1|B_+) \times \Pr(T_1|C_+) \times \Pr(A_+ \cap B_+ \cap C_+)}{\Pr(T_1)} \\ &= 0.504 \times \frac{\Pr(A_+ \cap B_+ \cap C_+)}{\Pr(T_1)}\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Pr(A_+ \cup B_+ \cup C_+ \mid T_1) &= \frac{\Pr(T_1|A_+ \cup B_+ \cup C_+) \times \Pr(A_+ \cup B_+ \cup C_+)}{\Pr(T_1)} \\ &= \frac{(1 - \Pr(T_1|A_- \cap B_- \cap C_-)) \times \Pr(A_+ \cup B_+ \cup C_+)}{\Pr(T_1)} \\ &= \frac{(1 - 0.1 \times 0.2 \times 0.3) \times \Pr(A_+ \cup B_+ \cup C_+)}{\Pr(T_1)} \\ &= 0.994 \times \frac{\Pr(A_+ \cup B_+ \cup C_+)}{\Pr(T_1)}\end{aligned}$$

Part c and d asks for $\Pr(T_1|C_1)$ where C_1 corresponds to the event that the item is of Class 1. Contrary to part a and b above, $\Pr(T_1|C_1)$ can be computed explicitly. For part c, we have

$$\Pr(T_1|C_1) = \Pr(T_1|A_+ \cap B_+) = \Pr(T_1|A_+) \times \Pr(T_1|B_+) = 0.9 \times 0.8 = 0.72$$

Similarly, we also have for part d

$$\begin{aligned}\Pr(T_1|C_1) &= \Pr(T_1|A_+ \cap B_+ \cap C_-) + \Pr(T_1|A_+ \cap B_- \cap C_+) \\ &\quad + \Pr(T_1|A_- \cap B_+ \cap C_+) + \Pr(T_1|A_+ \cap B_+ \cap C_+) \\ &= 0.9 \times 0.8 \times 0.3 + 0.9 \times 0.2 \times 0.7 + 0.1 \times 0.8 \times 0.7 + 0.9 \times 0.8 \times 0.7 = 0.902\end{aligned}$$

If we now reverse the wording of the problem so that the values in the above table represent the probability $\Pr(X|T_1)$ where $X \in \{A_+, A_-, B_+, B_-, C_+, C_-\}$, then the modified part a and part b are easy to solve using the technique that solves part c and part d in the original problem, while the modified part c and part d are then similar to part a and part b in the original problem. Their solutions will be omitted here.

3. (20 points) Suppose Algorithm 1.8 is modified to multiply an m digit number by an n digit number, where each digit occupies one computer word.

   (a) (10 points) As a function of m and n, how many one digit numbers are multiplied ?

   (b) (10 points) How many times is a two digit number, a one digit number, and a carry added ?

   Solution: We will represent the two numbers as X and Y, where

$$\begin{aligned}X &= x_{m-1}b^{m-1} + x_{m-2}b^{m-2} + \ldots + x_1 b + x_0 \\ Y &= y_{n-1}b^{n-1} + y_{n-2}b^{n-2} + \ldots + y_1 b + y_0\end{aligned}$$

Algorithm 1.8 will then be modified to loop through the index i from 0 to $m-1$ in the outerloop and to loop through the index j from 0 to $n-1$ in the inner loop. For both part a and b, it's easy to see that the number of operations done

```
1.1  for i ← 0 to m + n − 1 do z_i ← 0;
1.2  for i ← 0 to m − 1 do
1.3  |   c ← 0 ;
1.4  |   for j ← 0 to n − 1 do
1.5  |   |   p ← x_i y_j + z_{i+j} + c ;
1.6  |   |   z_{i+j} ← p  mod b ;
1.7  |   |   c ← ⌊p/b⌋;
1.8  |   end
1.9  |   z_{i+n} ← c
1.10 end
```

Algorithm 1: Integer multiplication

is just the number of time line 1.5 is done. Since the outerloop is done m times, and for each time the outerloop is done, the inner loop is done n, times. Therefore, the total number of times line 1.5 is done is mn times. So the number of multiplications done is mn, and the number of additions of a two-digit numbers $x_i y_j$, a one-digit numbers z_{i+j} and a carry c is added is also at most mn (it could potentially be less then mn since $x_i y_j$ might not always be a two-digit numbers.)

4. (20 points) Suppose Algorithm 1.8 is modified to multiply an m/s digit number by an n/s digit number, where each digit needs s words. Also assume that m/s and n/s are integers.
   (a) (10 points) How many s word numbers are multiplied by the algorithm ?
   (b) (10 points) How many times is a $2s$ word number, a s word number and a carry added ?

   Solution We will represent the two numbers as X and Y where

$$X = x_{m/s-1}b^{m/s-1} + x_{m/s-2}b^{m/s-2} + \ldots + x_1 b + x_0$$
$$Y = y_{n/s-1}b^{n/s-1} + y_{n/s-2}b^{n/s-2} + \ldots + y_1 b + y_0$$

   where each digit b occupy s words. Again, Algorithm 1.8 is modifed to loop through the index i from 0 to $m/s-1$ in the outerloop and to loop through the index j from 0 to $n/s-1$ in the inner loop. By the same analysis as in Question 3, we easily see that the number of multiplications done is $m/s \times n/s = mn/s^2$ and the number of additions of a $2s$ word numbers $x_i y_j$, a s words number z_{i+j} and a carry c is done is also at most mn/s^2.

5. (20 points) Suppose an m word number is multiplied by an n word number using the algorithm of Question 4, where that algorithm calls the algorithm of Question 3 to do its multiplication of s word numbers. Assume also that m and n are multiples of s.
   (a) (5 points) How many times will this combined algorithm multiply one word numbers ?
   (b) (5 points) How many times will this combined algorithm perform the additions referred to in Question 3b ?
   (c) (5 points) How many times will this combined algorithm perform the additions referred to in Question 4b ?
   (d) (5 points) Let t_a be the time needed to multiply one word numbers, t_b be the time needed to do the additions referred to in Question 3b, and t_c be the time needed to do the additions referred to in Question 4b. Write three formulas for the following cases
      i. The time needed to multiply an m word number by an n word number using the algorithm of Question 3 directly
      ii. The time needed to multiply the same numbers using the combined algorithm of this question.
      iii. The result of subtracting the time from part i from the time from part ii.

      Finally say which algorithm is faster, or discuss the situation if it's not entirely clear which method is faster.

   Solution: The combined algorithm multiply an m words number by an n words number by splitting the m words number into m/s digits with each digits being s words as well as splitting the n words number into n/s digits, again with each digits being s words. By Question 4a, the number of s words being multiplied is then mn/s^2. However, Question 5a asks for the number of one words being multiplied. Since each s word numbers can be view as s digits with each digits of size one words, by Question 3a, the number of one words multiplication done when multiplying an s words number by an s word numbers is then s^2. Therefore, the number of one words multiplication done by the combined algorithm is $mn/s^2 \times s^2 = mn$. By similar reasoning, at most mn number of additions referred to in Question 3b is done. For Question 5c, only mn/s^2 number of additions referred to in Question 4b is done.

   Now, for 4d(i), the formula is $mn(t_a + t_b)$ since from the reasoning for Question 3a and 3b, mn multiplication of one words number is done and mn additions referred to in Question 3b is done. Since each multiplication takes time t_a and each additions take time t_b, the total time is $mn(t_a+t_b)$. For 4d(ii), the total time is $mn(t_a+t_b)+mn/s^2 \times t_c$ since the combined algorithm have to do the multiplication of one words number, the additions referred to in Question 3b, and the additions referred to in Question 4b, and from the above reasoning, we know that the number of times the above operations need to be done is mn, mn, and mn/s^2, respectively. Therefore, the total time is $mn(t_a+t_b+t_c/s^2)$. Subtraction of the time for 4d(i) from the time for 4d(ii) then gives mnt_c/s^2. Therefore, the combined algorithm is slower since apart from all the additions and multiplications that's also done by the algorithm in Question 3, the combined algorithm of this question also have to do the mn/s^2 additions referred to in Question 4b.