Final	Purdom
12-20-02	B403

1. Solve for z.

1a.

$$\ln(2) = \ln(1 + 1/2) + \ln(1 + z).$$

1b.

$$\ln(2) = \ln(1 + 1/4) + \ln(1 + 1/5) + \ln(1 + 1/6) + \ln(1 + 1/7) + \ln(1 + z).$$

1c.

$$\ln(2) = \ln[(1 + z)/(1 - z)].$$

1d.

$$\ln(2) = \ln[(1 + z)/(1 - z)] + \ln[(1 + w)/(1 - w)],$$

where $w = 1/5$.

2. Give the Taylor series with a remainder for

2a.

$$\ln(1 + x).$$

2b.

$$\ln\left(\frac{1 + x}{1 - x}\right).$$

For the following problems, when asked to compute an upper limit on the time needed to do a computation, you should neglect the time for any parts of the problem other than big number arithmetic. Make the following assumptions about the time needed to do integer arithmetic when the first number has m digits and the second one has n digits:

1. Addition or subtraction: $A \max\{m, n\}$.
2. Multiplication or division: Mmn.
3. Greatest common divisor: $Gmn \min\{m, n\}$.

Assume that each digit can represent a number in the range 0 to $2^{32} - 1$.

On question where you are asked to compute a time, find exact expressions when it is not too hard to do so, but use simple upper bounds if the math starts to become too complex. The remaining questions assume that you have routines available to do integer arithmetic, but that you don't have routines available to do fraction arithmetic. You will need to reduce each problem in fraction arithmetic down to a set of problems in integer arithmetic, as you did in grade school, but this time you are expected to be clever.

3. Suppose you compute
$$N_k = 1$$
and
$$D_k = D_{k-1} y,$$
where $D_0 = 1$ and y is an integer.
   - 3a. How many digits are needed to represent y? (Your answer will probably be a formula containing $\lg y$.)
   - 3b. What is the value of D_k? How many digits are needed to represent D_k?
   - 3c. How long does it take to compute D_k.
   - 3d. How long does it take to compute all the D_k for $0 \le k \le n$?
4. Suppose you compute
$$\frac{A_k}{B_k} = \frac{A_{k-1}}{B_{k-1}} + \frac{N_k}{D_k}$$
with $A_0 = 1$ and $B_0 = 1$, where N_k and D_k come from the problem 3 (this is addition of fractions). This would completely define A_k and B_k if we required that the fractions be in lowest terms. However, some parts of the problem do not have that restriction, leading to many possible values for A_k and B_k.
   - 4a. Show that $B_k = D_k$ is a possible solution to these equations with B_k being an integer.
   - 4b. Give a recurrence for the A_k that matches the B_k in question 4a, and prove that the resulting A_k is always an integer.
   - 4c. Prove that the solution to problems 4a and 4b gives a result that is in lowest terms.
   - 4d. What is a good bound on the time needed to find all the A_k and B_k for $0 \le k \le n$.
5. One of the clever ways of doing the project leads to the following calculation:
$$\frac{A_k}{B_k} = \frac{A_{k-2}}{B_{k-2}} + \frac{N_k}{kD_k}$$
with $A_1 = 1$ and $B_1 = y$, where N_k and D_k come from the problem 3. (You don't need to know this to work the question, but the y in question 3 is related to x^2 from question 2b.) The early subquestions will be graded both on correctness and the extent they lead toward a good solution to 5d.
   - 5a. Give a good set of numbers to use for the B_k. Question 4a gave a perfect set of denominators of its question. Probably there is no perfect set of denominators for this question, but using ideas from 4a, you can probably come up with a good set of denominators. The simplest answer to this part is have $B_k = B_{k-2} D_k$, but this choice will lead to a poor final result for question 5d.
   - 5b. Give a recurrence for the A_k that matches the B_k in question 5a.
   - 5c. Show how to compute A'_k and B'_k in lowest terms. The simplest answer to this question is $A'_k = A_k / GCD(A_k, B_k)$ and $B'_k = B_k / \text{GCD}(A_k, B_k)$, but this will lead to a poor answer for part 5d.
   - 5d. For the method you developed in 5a, 5b, and 5c, what is a good bound on the time needed to find all the A'_k and B'_k for $0 \le k \le n$.