

1. Consider an integer in base b with n digits where the leading digit is not zero.
 - 1a. What is the smallest such integer?
 - 1b. What is the largest such integer?
 - 1c. If all such integers are equally likely what is the probability of the integer where all the digits are equal to $b - 1$?
 - 1d. If two such integers are chosen at random (each possible integer being equally likely), what is the probability that their sum needs $n + 1$ digits.
2. Consider the problem of efficiently adding an m digit base b number (no leading zero) to an n digit base b number (no leading zero), where you are permitted to store the answer on top of the n digit number. Assume $n \geq m$.
 - 2a. Give an efficient algorithm for the problem. (Algorithm 1.4 might be a useful starting point.)
 - 2b. Give an average time analysis of your algorithm when the input numbers are random in the sense of Question 1.
3. Consider a potential algorithm that is going to multiply three-digits numbers based on additions and multiplications of one-digit numbers. The intended algorithm will work with any positive integer base so that it can be used recursively (just as algorithm 5.2 is used).
 - 3a. Let k be the number of one-digit numbers used by the algorithm. What is the largest value for k such that the potential algorithm is more efficient than Algorithm 5.2 for large number?
 - 3b. Assume that the running time of the algorithm obeys the recurrence

$$T(n) = kT(n/3) + an + b.$$

What is the running time when n is a power of three? In your final answer, be sure to use the value of k from Question 3a and give the answer in the simplest form.

4. This question ask you to modify Algorithm 5.2 so that U has m bits, V has n bits, U_2 has k bits, and V_2 has k bits.
 - 4a. Give the code for an efficient algorithm based on these ideas.
 - 4b. Analyze the running time for the algorithm. Indicate clearly for each term in your running time formula which part of the algorithm is associated with it. In your formulas, please use the following notation: $M(m, n)$ is needed to multiply an m digit number by an n digit number; $A(m, n)$ is the time needed to add (or subtract) an m digit number to (from) an n digit number. To reduce your work (while having uniform grading) you are required to make the following simplifying assumptions:
 - A. No time is needed for anything expect addition and multiplication (thus, each term in your formula will have an A or an M in it).
 - B. Any sum or difference always leads to a result where the number of digits for the answer is equal to the number of digits for the larger input (no carries off the upper end, no borrows that reduce the number of digits).
 - C. Any products always lead to results where the number of digits for the answer is equal to the sum of the number of digits for each input.
5. Simplify your answer to Question 4 by omitting the addition time. Also eliminate occurrence of max and min by doing a case analysis. In particular, give the simplified formula for each case below. For each case, also give the value of k that makes the multiplication-only time as small as possible. Clearly indicate which case gives the optimum value for k . Finally, give the optimum value of k .
 - 5a. $2k \leq m \leq n$.
 - 5b. $m \leq 2k \leq n$.
 - 5c. $m \leq n \leq 2k$.