

1. Write an algorithm to find an optimum solution for the following problem. Your input is an array of numbers $C[i]$ for $0 \leq i \leq k$. The number $C[i]$ is the number of dollars that a gallon of gas cost at mile i . You want to drive from mile 0 to mile k . Your car starts with no gas. Its tank holds 20 gallons of gas. Your car needs one gallon of gas for each mile (it uses a lot of gas). Your program must fill in an array $B[i]$ that says how much gas to buy at mile i . Clearly, you must have

$$\sum_{0 \leq j < i} B[j] \geq i$$

so that you don't run out of gas. Also you must have

$$\sum_{0 \leq j \leq i} B[j] \leq i + 20$$

so that you have room to keep all the gas you buy. You want the total cost to be as low as possible.

Hint. Clearly, you want to have no gas left at the end of the trip. Therefore, it is easy to figure out where to buy the last gas and how much to buy.

Alternate hint. Clearly, you must buy some gas at the start of the trip. It is easy to figure out how much gas to buy at the first place.

The ideal algorithm for this problem runs in linear time.

2. Write a fast algorithm for the following problem. Your input is an array of characters $C[i]$ for $1 \leq i \leq n$ and an integer k . Your output is an array of Booleans $B[i]$ for $k \leq i \leq n$. You must set $B[i]$ to *true* if the number of 'A's in $C[i - k + 1]$ to $C[i]$ is at least $k - 20$, and to *false* otherwise.

The ideal algorithm for this problem runs in time proportional to n and independent of k .

3. Suppose you program Strassen's Algorithm so that the running time obeys the recurrence

$$T_n = 7T_{n/2} + an^2 \quad \text{nanoseconds} \quad \text{for } n > b,$$

$$T_n = n^3 \quad \text{nanoseconds} \quad \text{for } n \leq b.$$

- a. What value of b (as a function of a), should you use to obtain the best running time.
- b. What is the running time (when the best b is used)?

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- a. What value of b (as a function of a), should you use to obtain the best running time?
 - b. What is the running time (when the best b is used)?
5. Compute the number of unrooted trees where each node has degree one (leaves) or three (internal nodes) as a function of the number of nodes on degree one. Assume that the leaves are labelled, but the internal nodes are not.

Hints: There is one tree with two nodes of degree one. (Have an edge between the two nodes. For any tree with n nodes of degree one, you can form a tree with $n + 1$ nodes of degree one having an edge from the new node to the middle of an edge of the old tree. This results in one new node of

degree one (the node that was added) and one new node of degree three (where the edge from the new node joins the edge from the old tree). Since the edges have (distinct) labels, each of these new trees is different from each other. Each different n node starting tree also leads to different $n + 1$ node trees.

6. Suppose you have m distinct objects and n distinct names. How many different ways can you give each object a unique name.

Example: If you have objects 1 and 2 and names a , b , and c , then there are 6 ways ($[1 : a, 2 : b]$, $[1 : a, 2 : c]$, $[1 : b, 2 : a]$, $[1 : b, 2 : c]$, $[1 : c, 2 : a]$, $[1 : c, 2 : b]$).

7. What is the general solution to the recurrence

$$X_i = -X_{i-1} + 2X_{i-2}?$$

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$$X_i = X_{i-1} + 2X_{i-2}?$$

9. Suppose you want to compute e , the base of natural logarithms to ten significant figures, using only pencil and paper. The book should suggest several ways of doing this. Try to pick one that is reasonably efficient and explain how much work it would take. Note that this question only asks you to explain how to do the calculation and the amount of work you would need to do. It does not ask you to do the work.
10. Suppose you want to compute $\ln 2$, the natural logarithm of two to ten significant figures, using only pencil and paper. The book should suggest several ways of doing this. Try to pick one that is reasonably efficient and explain how much work it would take. Note that this question only asks you to explain how to do the calculation and the amount of work you would need to do. It does not ask you to do the work.
11. Simplify $\sum_i i^2 \binom{n}{i}$.
12. Simplify $\sum_i i^2 \binom{n}{i} x^i$.