

1. Suppose X is a random integer in the range 0 to $b^n - 1$, and suppose X is written with n digits in base b .
 - 1a. What is the probability that the most significant digit (the uppermost of the n digits) of X is 0?
 - 1b. What is the probability that the most significant digit of X is 1?
2. Suppose X and Y are independently selected random integers, each in the range 0 to $b^n - 1$, and suppose $Z = X + Y$. Assume Z is written with $n + 1$ digits in base b .
 - 2a. What is the probability that the most significant digit of Z is 0?
 - 2b. What is the probability that the most significant digit of Z is 1?
3. Find an asymptotic solution to the equation $x = t + x^{1/5}$ for large t .
4. A student, after thinking carefully about the problem of multiplying integers wrote a classical subroutine that could multiply an m digit integer by an n digit integer in time

$$6 + 9n + 9mn.$$

With additional thought, he developed a Karatsuba style subroutine that could run in time

$$6 + 4 \max\{i, m - i\} + 4 \max\{i, n - i\} + T(i, i) + T(\max\{i, m - i\}, \max\{i, n - i\}) + T(m - i, n - i)$$

where $i = \lfloor n/2 \rfloor$ and $T(m, n)$ is the time needed by the best of these two subroutines for multiplying an m digit integer by an n digit integer. For which values of m and n should this student use that classical subroutine and for which values should he use the other one?

5. With yet more thought, the student (from question 4) realized that his Karatsuba style code would work correctly for any value of i in the range $1 \leq i \leq \min\{m, n\}$ and that it would need the time already given in question 4. He also developed two recursive classical style algorithms. The first one needed time

$$6 + 4(n + m) + T(i, j) + T(m - i, j) + T(i, n - j) + T(m - i, n - j)$$

for any i and j in the ranges $1 \leq i \leq m$, $1 \leq j \leq n$ and the second one needed time

$$6 + 4m + T(i, n) + T(m - i, n)$$

for any i in the range $1 \leq i \leq m$. In these equations, each $T(m, n)$ is the least time needed to multiply an m digit integer by an n digit integer using any of these available algorithms. (Clearly, which algorithm is fastest will depend on the values of m and n , so the various T s may be referring to different algorithms.) The student also noticed that by carefully choosing the order of the arguments his subroutines, he could obtain the times implied by interchanging m and n in the above expressions. Fortunately, the student also knew how to apply dynamic programming to the problem.

- 5a. Write down the equations whose solution will give the time needed by the best program that combines the above ideas in the best possible way. Note that you only need to write down the equations, you don't need to solve them.
- 5b. Explain how to solve in polynomial time the equations that you wrote down as your answer for 5a. (If your equations can not be solved in polynomial time, then you need some easier to solve equations for 5a.) Also, give a rough analysis (big O on the leading term) of the time needed to solve your equations. Again, note that you only need to explain how to solve your equations quickly; you don't need to solve them.