Solution to problem 4a.

The probability that the first integer is j is $1/n$.

The probability that the first i integers are less than j is

$$\frac{j-1}{n-1}\frac{j-2}{n-2}\cdots\frac{j-i}{n-i} = \frac{(j-1)!(n-i-1)!}{(j-i-1)!(n-1)!} = \frac{\binom{j-1}{i}}{\binom{n-1}{i}}.$$

(There are i terms in the product.)

The probability that the remaining $k-i$ integers are greater than j is

$$\frac{n-j}{n-i-1}\frac{n-j-1}{n-i-2}\cdots\frac{n+i-j-k+1}{n-k} = \frac{(n-j)!(n+i-j-k)!}{(n+i-j-k)!(n-i-1)!} = \frac{\binom{n-j}{k-i-1}}{\binom{n-i-1}{k-i-1}}.$$

(There are $k-i-1$ factors.)

Since we don't care which order the lesser and greater integers are in, we need to multiply these three factors by $\binom{k-1}{i}$ giving

$$p_{ijkn} = \frac{1}{n}\frac{\binom{k-1}{i}\binom{j-1}{i}\binom{n-j}{k-i-1}}{\binom{n-1}{i}\binom{n-i-1}{k-i-1}}.$$

Partial check of the formulas. Suppose $n=3$, $j=2$, $k=3$, $i=1$.

The probability that the first number is 2 is 1/3.

The probability that the second number is 1 is 1/2. (The second number must be 1 or 3 since the first number is 2).

The probability that the third number is 3 is 1.

The other way that we can have one number on the side that i is counting is to have the selected numbers be 2, 3, 1 in that order. This has the same probability as the previous case. Thus the total probability is 2 times each case, in keeping with $\binom{2}{1} = 2$.

The total probability by direct calculation is $(1/3)(1/2)(1)(2) = 1/3$.

The total probability by the claimed formula is

$$\frac{1}{3}\frac{\binom{2}{1}\binom{1}{1}\binom{1}{1}}{\binom{2}{1}\binom{1}{1}} = \frac{1}{3}\frac{2\cdot 1\cdot 1}{2\cdot 1} = \frac{1}{3}.$$