Test 1 Purdom
3-2-09 B503 Analysis of Algorithms

All answers should be as simple as possible.

1. Suppose you flip a fair coin one or more times, where you continue flipping as long as the previous flip resulted in heads. You stop as soon as a flip results in tails.

   a. What is the probability that there will be exactly i flips?

   b. What is the average number of flips?

2. Suppose you have a box with width a, depth b, and height c. Consider the set of all paths made up of unit length steps, where a single step can go one unit in increasing width, one unit in increasing depth, or one unit in increasing height. How many paths are there from the origin, to the far top right corner? If we describe the box in coordinates, we are asking how many paths are there from the corner at $(0, 0, 0)$ to the corner at (a, b, c)?

3. Suppose that integer A is equally likely to have any value in the range $0 \leq A \leq b^m - 1$, integer B is equally likely to have any value in the range $b^{m-1} \leq B \leq b^m - 1$, integer C is equally likely to have any value $0 \leq C \leq b^n - 1$, and integer D is equally likely to have any value in the range $b^{n-1} \leq D \leq b^n - 1$. (All numbers are assigned their values independently.) Suppose $m = n$.

   a. What is the probability that $A + 1 \geq b^m$?

   b. What is the probability that $B + 1 \geq b^m$?

   c. What is the probability that $A + C \geq b^m$?

   d. What is the probability that $B + C \geq b^m$?

   e. What is the probability that $B + D \geq b^m$?

4. Suppose that B and D are the same as question 3, but $n > m$. Suppose we add B and D using base-b arithmetic the standard algorithm (similar to Algorithm 1.4, but with changes needed to match this question).

   a. What is the probability that there is a carry from position m to position $m + 1$?

   b. What is the conditional probability that there is a carry from position i to position $i + 1$ (with $i > m$) under the condition that there was a carry from position $i - 1$ to position i?

   c. What is the probability that there is a carry from position i to position $i + 1$ (with $i > m$)?

5. Let $p(i) = \binom{n}{i} 2^{-n}$.

   a. What is the average value of $p(i)$ (as i ranges over all possible values)?

   a. What is the variance of $p(i)$?