# C241 Fall 08: Final Exam Review

**1) Use the language P defined below to answer the following questions.**

| | $P \subset V^+ V = \{a, b, c\}$ |
|---|---|
| 1. | $a \in P$ |
| 2a. | $u \in P \Rightarrow bub \in P$ |
| 2b. | $u \in P \Rightarrow ucu \in P$ |
| 3. | There is nothing else in P. |

**1a) Which of these strings are in the language P?**

| | | | |
|---|---|---|---|
| (i) | a | (v) | aca |
| (ii) | bab | (vi) | aacaa |
| (iii) | b | (vii) | ababcbaba |
| (iv) | bcb | (viii) | bacacacab |

**1b) Using structural induction proofs, prove that every element in P has the following properties:**

(i) Every $p \in P$ has an even number of $b$'s.

(ii) Every $p \in P$ has at least one $a$.

(iii) Every $p \in P$ has an odd number of letters.

(iv) Every $p \in P$ is a palindrome.

**2) Use the language L, and the functions I, A, and R defined below to answer the following questions.**

| | $L \subset V^+V = \{a, b, \dot{}\}$ |
|---|---|
| 1. | $\bullet \in L$ |
| 2a. | $u \in L \Rightarrow au \in L$ |
| 2b. | $u \in L \Rightarrow bu \in L$ |
| 3. | There is nothing else in L. |

| | $I : L \to L$ |
|---|---|
| I1. | $I(\bullet) = \bullet$ |
| I2a. | $I(au) = bI(u)$ |
| I2b. | $I(bu) = aI(u)$ |

| | $R : L \to L$ |
|---|---|
| R1. | $R(\bullet) = \bullet$ |
| R2a. | $R(au) = A(R(u), a\bullet)$ |
| R2b. | $R(bu) = A(R(u), b\bullet)$ |

| | $A : L \times L \to L$ |
|---|---|
| A1. | $A(\bullet, v) = v$ |
| A2a. | $A(au, v) = aA(u, v)$ |
| A2b. | $A(bu, v) = bA(u, v)$ |

| | $T : L \times L \to L$ |
|---|---|
| T1. | $T(\bullet, v) = v$ |
| T2a. | $T(au, v) = T(u, av)$ |
| T2b. | $T(bu, v) = T(u, bv)$ |

**2a) Label the following statements True or False:**

i) $\bullet \in L$

ii) $a \in L$

iii) $R(bb\bullet) = A(R(b\bullet), b\bullet)$

iv) $A(u, R(\bullet)) = \bullet$

v) $A(v, A(R(u), a\bullet)) = A(v, A(u, a\bullet))$

vi) $I(ab\bullet) = bI(a\bullet) = baI(\bullet) = ba\bullet$

vii) $I(u) = u \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix}$

viii) $I(u) = u \begin{bmatrix} a, & b \\ b, & a \end{bmatrix}$

**2b) Write a recursive function (in the style of the functions I, A, and R above) which takes a string $u \in L$ and performs the substitution $u \begin{bmatrix} bb \\ b \end{bmatrix}$.**

**2c) Prove that for all $u, v \in L$, $T(u, v) = A(R(u), v)$. In addition to the function definitions above, you can use the following three lemmas which were proved on the homework:**

For all $u, v, w \in L$, $A(A(u, v), w) = A(u, A(v, w))$

For all $u \in L$, $A(u, \bullet) = u$

For all $u, v \in L$, $R(A(u, v)) = A(R(v), R(u))$

**3) Answer the following questions about trees.**

3a) Draw all the non-isomorphic trees (not necessarily binary trees) of size 3.

3b) Draw all the non-isomorphic *binary* trees of depth 2.

**4) The language T which is defined below represents the set of binary trees. Use it to answer the following questions.**

| | $T \subset \{n, l, ), (\}^+$ |
|---|---|
| 1. | $l \in T$ |
| 2. | $t_1, t_2 \in T \Rightarrow (n\ t_1\ t_2) \in T$ |
| 3. | There is nothing else in T. |

**4a) Which of these strings are in the language T?**

| | | | |
|---|---|---|---|
| (i) | l | (iv) | (l) |
| (ii) | (n l l) | (v) | (n n l) |
| (iii) | (n (n l l) l) | (vi) | (n (n l l) (n l (n l l))) |

**4b) Fill in the blanks to complete the following recursive functions:**

| | $LeafCount : T \rightarrow N$ (LeafCount(t) = the number of $l$'s (leaves) in t) |
|---|---|
| L1. | $LeafCount(l) =$ _____ |
| L2. | $LeafCount(n\ t_1\ t_2) = LeafCount(t_1)$ _____ $LeafCount(t_2)$ |

| | $IntNodes : T \rightarrow N$ (IntNodes(t) = the number of $n$'s (interior nodes) in t) |
|---|---|
| N1. | $IntNodes(l) =$ _____ |
| N2. | $IntNodes(n\ t_1\ t_2) = IntNodes(t_1)$ _____ $IntNodes(t_2)$ _______________ |

**4c) Prove that for all $t \in B$, $LeafCount(t) = IntNodes(t) + 1$**

**5) Given $P \equiv x+(y-z)+wx$, write the results of the following substitutions:**

a) $P\begin{bmatrix} a, & b \\ x, & y \end{bmatrix}$

b) $P\begin{bmatrix} b \\ q \end{bmatrix}$

c) $P\begin{bmatrix} y, & b \\ x, & y \end{bmatrix}$

d) $P\begin{bmatrix} y \\ x \end{bmatrix}\begin{bmatrix} b \\ y \end{bmatrix}$

e) $P\begin{bmatrix} (y-z) \\ x \end{bmatrix}\begin{bmatrix} (y-z) \\ w \end{bmatrix}$

**6) Answer the following questions about the program verification problem below (x != 0 is another way of writing $x \neq 0$:**

```
{x = A}
z := B;
while (x != 0) {x + z = A + B}
   if  (x > 5)
       then begin  z := z + 5;  x := x - 5;  end
       else begin  x := x - 1;  z := z + 1; end
end
{z = A+B}
```

**6a) Is this an appropriate outline of the verification proof for this program? Correct any mistakes in the outline below:**

**Label the invariant:** $I \equiv \{x + z = A + B\}$

**Step 1: Prove I** $\left[ \begin{array}{cc} A, & B \\ x, & z \end{array} \right]$ **is true.**

**Step 2a: Prove** $I \wedge (x > 5) \Rightarrow I \left[ \begin{array}{cc} A - 5, & B + 5 \\ x, & z \end{array} \right]$

**Step 2b: Prove** $I \wedge (x \leq 5) \Rightarrow I \left[ \begin{array}{cc} x - 1, & z + 1 \\ x, & z \end{array} \right]$

**Step 3: Prove** $I \wedge (x \leq 5) \Rightarrow \{z = A + B\}$

**6b) Complete the verification proof for the program above.**

**7) Prove that for all $n \geq 0$, $t_n = a + \frac{n^2+n}{2}$, given the following recursive definition of $t_i$:**

$$t_0 = a$$

$$\vdots$$

$$t_{k+1} = t_k + k + 1$$