C241 Homework Assignment 10

1. The language L and functions R, A, and T, defined below, are the same as in Section 7.7.

$$
\begin{array}{ll}
 & \underline{L \subseteq \{\texttt{a},\ \texttt{b},\ \bullet\}^+} \\
1. & \bullet \in L \\
2. & u \in L \Rightarrow \begin{cases} \texttt{a}u \in L \\ \texttt{b}u \in L \end{cases} \\
3. & n.\ e.
\end{array}
$$

	$A\colon L^2 \to L$	$R\colon L \to L$	$T\colon L^2 \to L$
1.	$A(\bullet, v) = v$	$R(\bullet) = \bullet$	$T(\bullet, v) = v$
2a.	$A(\texttt{b}u, v) = \texttt{b}A(u, v)$	$R(\texttt{a}u) = A(R(u), \texttt{a}\bullet)$	$T(\texttt{a}u, v) = T(u, \texttt{a}v)$
2b.	$A(\texttt{a}u, v) = \texttt{a}A(u, v)$	$R(\texttt{b}u) = A(R(u), \texttt{b}\bullet)$	$T(\texttt{b}u, v) = T(u, \texttt{b}v)$

Prove the following:

(a) *For all* u, v, $w \in L$, $A(A(u, v), w) = A(u, A(v, w))$.

(b) *For all* $u \in L$, $A(u, \bullet) = u$.

(c) *For all* u, $v \in L$, $R(A(u, v)) = A(R(v), R(u))$.

(d) *For all* u, $v \in L$, $T(u, v) = A(R(u), v)$.

(e) *For all* $u \in L$, $R(u) = T(u, \bullet)$.

2. Let $P \equiv p \wedge (q \vee r)$. Perform the following substitutions

(a) $P \begin{bmatrix} r, q, p \\ p, q, r \end{bmatrix}$

(b) $P \begin{bmatrix} p \vee r \\ p \end{bmatrix}$

(c) $P \begin{bmatrix} p \vee r, & q \Rightarrow r \\ p, & r \end{bmatrix}$

(d) $\left(P \begin{bmatrix} p \vee r \\ p \end{bmatrix} \right) \begin{bmatrix} q \\ p \end{bmatrix}$

(e) $\left(P \begin{bmatrix} q \\ p \end{bmatrix} \right) \begin{bmatrix} p \vee r \\ p \end{bmatrix}$

3. Are the following equivalences valid? If not, give a counterexample.

(a) $F\left[{A,\ B \atop x,\ y}\right] \stackrel{?}{\equiv} F\left[{B,\ A \atop y,\ x}\right]$

(b) $F\left[{A,\ B \atop x,\ y}\right] \stackrel{?}{\equiv} F\left[{A \atop x}\right]\left[{B \atop y}\right]$

(c) $F\left[{A \atop x}\right]\left[{B \atop y}\right] \stackrel{?}{\equiv} F\left[{B \atop y}\right]\left[{A \atop x}\right]$

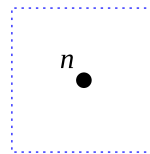
(d) $F\left[{\left[{B \atop y}\right] \atop x}\right] \stackrel{?}{\equiv} F\left[{A \atop x}\right]\left[{B \atop y}\right]$

4. Recall that a *tree*, $T \subseteq N \times N$ is a finite acyclic graph in which one node $r \in N$ has in-degree 0 and all other nodes have in-degree 1. A node with out-degree 0 is called a *leaf*.

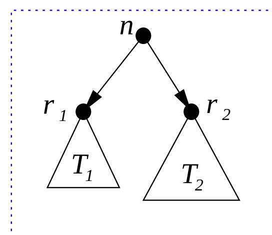
Definition. A *binary tree* is a tree in which all nodes which are not leaves have out-degree 2.

Let us define a set $\mathcal{B}$ of graphs inductively as follows:

(1) If $n \in N$ then $\emptyset \subset \{n\} \times \{n\}$ is and element of $\mathcal{B}$ and has root n.



(2) If $R_1 \cap R_2 = \emptyset$, and $T_1 \in R_1 \times R_1$ and $T_2 \in R_2 \times R_2$ are elements of $\mathcal{B}$ with roots r_1 and r_2, then for any $n \notin R_1 \cup R_2$, the relation $\{(n, r_1), (n, r_2)\} \cup T_1 \cup T_2$ is an element of $\mathcal{B}$ with root n.



(3) Nothing else is in $\mathcal{B}$.

Problems:

(a) Prove: *Every relation $R \in \mathcal{B}$ is a binary tree.*

(b) Prove: *For any tree $T \in \mathcal{B}$, the number of nodes with out-degree 0 is exactly one greater than the number of nodes with out-degree 2.*

(c) The *depth* of any tree is defined to be the length of the longest path from its root to a leaf. Define a recursive function $depth \colon \mathcal{B} \to \mathbb{N}$ that gives the depth of a tree.

(d) Prove: *For any tree $T \in \mathcal{B}$ of depth d, the number of its nodes is no greater than $2 \cdot 2^d$.*

5. The program is the same as the one on Test Two. Prove the verification conditions using Proposition 8.7 (Substitution) to handle assignment statements, as is illustrated in Example 8.8.

$\{x = A \wedge y = B\}$
$z := 0$;
`while` $y \neq 0$ $\{z + x \cdot y = A \cdot B\}$
  `if` $even?(y)$
  `then begin` $y := y \div 2$; $x := 2 \cdot x$ `end`
  `else begin` $y := y - 1$; $z := z + x$ `end`
$\{z = A \cdot B\}$

6. Supplemental Problem. *This is a good test of whether you understand Chapter 7. Hint: N is intended to model the natural numbers, with P representing addition and M representing multiplication. Before attempting the proofs it is worthwhile to figure out what they are actually saying in terms of $\mathbb{N}$.*

For the following problems, consider the language $N \subseteq \{\mathtt{S}, \bullet\}$, defined inductively as follows:

1. $\bullet \in N$
2. $u \in N$ implies $\mathtt{S}u \in N$
3. nothing else

Define the function $P\colon N^2 \to N$ recursively, according to:

1. $P(\bullet, v) = v$
2. $P(\mathtt{S}u, v) = \mathtt{S}P(u, v)$

Define the function $M\colon N^2 \to N$ recursively, according to:

1. $M(\bullet, v) = \bullet$
2. $M(\mathtt{S}u, v) = P(v, M(u, v))$

Prove the following:

(a) *M distributes over P; that is, for all $u, v, w \in N$, $M(u, P(v, w)) = P(M(u, v), M(u, w))$.*

(b) *P is commutative; that is, For all $u, v \in N$, $P(u, v) = P(v, u)$.*

(c) *For all $u, v \in N$, $M(u, \mathtt{S}v) = P(u, M(u, v))$.*

(d) *For all $u, v \in N$, $P(\mathtt{S}u, v) = P(u, \mathtt{S}v)$.*

(e) *For all $u \in N$, $P(u, \bullet) = u$.*