

C241 Homework Assignment 1

1. (Exercise 1.1-1) List the following sets:

- (a) $\{2^i \mid i \in \mathbb{N} \text{ and } 0 \leq i \leq 8\}$
- (b) $\{i^2 \mid i \in \mathbb{N} \text{ and } 0 \leq i \leq 8\}$
- (c) $\{2k + 1 \mid k \in \mathbb{N}\}$
- (d) $\{m \mid 23 < m < 29 \text{ and } m \text{ is a prime number}\}$

SOLUTION

- (a) $\{1, 2, 4, 8, 16, 32, 64, 128, 256\}$
- (b) $\{0, 1, 4, 9, 16, 25, 36, 49, 64\}$
- (c) $\{1, 3, 5, 7, 9, \dots\}$
- (d) \emptyset

2. (Exercise 1.1-2) Let $A = \{a, b\}$; let $B = \{1, 2, 3\}$; let $C = \emptyset$; and let $D = \{a, b, c, d\}$. List the following sets:

- | | |
|--------------------------|------------------------------|
| (a) $A \cup B$ | (f) $A \cup C$ |
| (b) $A \cap B$ | (g) $A \cap D$ |
| (c) $A \times B$ | (h) A^3 |
| (d) $\mathcal{P}(A)$ | (i) $\mathcal{P}(\emptyset)$ |
| (e) $B \times \emptyset$ | (j) $(D \cap A) \times B$ |

SOLUTION

- (a) $\{a, b, 1, 2, 3\}$
- (b) \emptyset
- (c) $\{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
- (d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- (e) \emptyset
- (f) $\{a, b\}$
- (g) $\{a, b\}$
- (h) $\{(a, a, a), (a, a, b), (a, b, a), (a, b, b), (b, a, a), (b, a, b), (b, b, a), (b, b, b)\}$
- (i) $\{\emptyset\}$
- (j) $\{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

3. (Exercise 1.1-3) Let $A = \{a, b\}$; let $B = \{1, 2, 3\}$; and let $E = A \times B$. List the following sets:

(a) $\{(x, y, y) \mid (x, y) \in E\}$

(b) $\{(x, x) \mid x \in E\}$

(c) $\{(y, z) \mid (x, y) \in E \text{ and } z \in B\}$

SOLUTION

(a) $\{(a, 1, 1), (a, 2, 2), (a, 3, 3), (b, 1, 1), (b, 2, 2), (b, 3, 3)\}$

(b) $\{((a, 1), (a, 1)), ((a, 2), (a, 2)), ((a, 3), (a, 3)), ((b, 1), (b, 1)), ((b, 2), (b, 2)), ((b, 3), (b, 3))\}$

(c) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

4. Let $A = \{1, 2, 3, 4\}$. List the set $B = \{a + b \mid (a, b) \in A \times A\}$.

SOLUTION

$$A \times A = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), \\ (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$$

so

$$\begin{aligned} B &= \{(1 + 1), (1 + 2), (1 + 3), (1 + 4), (2 + 1), (2 + 2), (2 + 3), (2 + 4), \\ &\quad (3 + 1), (3 + 2), (3 + 3), (3 + 4), (4 + 1), (4 + 2), (4 + 3), (4 + 4)\} \\ &= \{2, 3, 4, 5, 3, 4, 5, 6, 4, 5, 6, 7, 5, 6, 7, 8\} \\ &= \{2, 3, 4, 5, 6, 7, 8\} \end{aligned}$$

You don't need to write all this out.

5. List the first twelve elements of the set

$$F = \{f_k \in \mathbb{N} \mid f_1 = 1, f_2 = 1, \text{ and for any } k \geq 1, f_{k+2} = f_k + f_{k+1}\}$$

SOLUTION

It is reasonable to assume (guess) that the subscript, i , is a simple index used in the typical way. So assume that $i \in \text{Whole}$ and the set F lists as

$$F = \{f_1, f_2, f_3, f_4 \dots\}$$

Elements f_1 and f_2 are given in the description,

$$F = \{1, 1, f_3, f_4, \dots\}$$

The description specifies a formula for the rest of the f_i s. $f_3 = f_1 + f_2 = 1 + 1 = 2$, so

$$F = \{1, 1, 2, f_4, f_5, \dots\}$$

and similarly $f_4 = f_2 + f_3 = 1 + 2 = 3$, and so on.

$$F = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots\}$$

This is a perfectly reasonable answer, but it's not quite "technically" right because $f_1 = f_2$. So we've listed the same element twice—which is OK—but the list accounts for only eleven elements. We need one more number to make twelve.

$$F = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \dots\}$$

DISCUSSION. *OK, it's something of a trick question aimed at getting you to think a bit more about the terminology of and notation for sets.*

By way of an apology FULL CREDIT TO EVERYONE that turned in an answer.

SUPPLEMENTAL PROBLEM. A small island is ruled by a benevolent queen. One morning, a proclamation is posted in the central common:

—To All My Subjects—

It has come to my attention that one or more husbands in my realm are unfaithful to their wives. I hereby decree that any woman who learns her husband is unfaithful must shoot him at the stroke of midnight.

Her Highness, The *QUEEN*

It's a small island. Every wife knows who all the unfaithful husbands are, *but does not know whether or not her own husband is faithful.*

On the third night, shots ring out. How many husbands are/were unfaithful?

SOLUTION

There were three unfaithful husbands, now all deceased.

To see why this is so, consider what would happen if there were:

1. just one unfaithful husband: *His wife knows:*

- *that all of all husbands except possibly her own are faithful.*
- *that there is at least one unfaithful husband because the Queen's proclamation says so.*

She must conclude her own husband is unfaithful (in fact, is the only unfaithful one), so at midnight of the first night, she would shoot him.

2. exactly two unfaithful husbands. *Their wives both know:*

- *There is at least one unfaithful husband, the one each knows about.*
- *There are at most two unfaithful husbands, if the wife's own husband is unfaithful.*
- *There are more than one unfaithful husbands, because no shots are heard on the first night.*

Both wives must conclude that their own husbands are unfaithful (and, in fact, that there are exactly two unfaithful husbands), so at midnight of the second night they shoot them.

n. In general, if a wife knows of n unfaithful husbands, she must wait until the n^{th} night to determine whether her own husband is unfaithful. If shots are heard, he is not; if no shots are heard, he is (and, in fact, there are exactly $n + 1$ unfaithful husbands in all). So at midnight on the $(n + 1)^{\text{st}}$ night, each of these $n + 1$ wives will shoot her husband.