

## C241 Homework Assignment 5

1. Prove that for all natural numbers,  $n$ ,

$$\sum_{i=0}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

SOLUTION

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The proof is by induction on  $k$  with hypothesis

$$H(k) \sum_{i=0}^k i^2 = \frac{k(k+1)(2k+1)}{6}$$

BASE CASE: To prove  $H(0)$ ,

$$0^2 = 0 = \frac{0 \cdot 1 \cdot 1}{6} = \frac{0 \cdot (0+1) \cdot (2 \cdot 0 + 1)}{6}$$

INDUCTION STEP: Assume  $H(k)$ . Now,

$$\begin{aligned} \sum_{i=0}^{k+1} i^2 &= \left( \sum_{i=0}^k i^2 \right) + (k+1)^2 && \text{(expand } \sum) \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{(IH)} \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} && \text{(multiply by } 1 = 6/6) \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{(add fractions)} \\ &= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} && \text{(factor } (k+1)) \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} && \text{(combine like terms)} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} && \text{(factor } 2k^2 + 7k + 6) \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} && \text{(arithmetic)} \end{aligned}$$

This completes the induction case and the proof of the problem.

2. Prove that for all natural numbers,  $n$ ,

$$\sum_{i=0}^n i(i!) = (n+1)! - 1$$

SOLUTION

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The proof is by induction on  $k$  with hypothesis

$$H(k) \equiv \sum_{i=0}^k i(i!) = (k+1)! - 1$$

Recall that

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ 1 \cdot 2 \cdot \dots \cdot n & \text{otherwise} \end{cases}$$

BASE CASE:

$$\sum_{i=0}^0 i(i!) = 0(0!) = 0 = 1 - 1 = 1! - 1 = (0+1)! - 1$$

INDUCTION STEP: Assume  $\sum_{i=0}^k i(i!) = (k+1)! - 1$ . Now,

$$\begin{aligned} \sum_{i=0}^{k+1} i(i!) &= \left( \sum_{i=0}^k i(i!) \right) + (k+1)(k+1)! && \text{(expand } \Sigma) \\ &= ((k+1)! - 1) + (k+1)(k+1)! && \text{(induction hypothesis)} \\ &= ((k+1)! + (k+1)(k+1)!) - 1 && \text{(rearrange sums)} \\ &= (1+k+1)(k+1)! - 1 && \text{distribute } (k+1)! \\ &= (k+2)(k+1)! - 1 && \text{(simplify)} \\ &= (k+2)! - 1 && \text{(meaning of '!')} \\ &= ((k+1)+1)! - 1 && \text{(as desired)} \end{aligned}$$

This proves  $H(k+1)$  and completes the induction.

3. Prove that for all natural numbers  $n$ , 6 evenly divides  $n^3 - n$ .

SOLUTION

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The proof is by induction on  $k \in \mathbb{N}$  with hypothesis  $H[n] \equiv$  "6 evenly divides  $n^3 - n$ "

BASE CASE ( $H[0]$ ).  $0^3 - 0 = 0$ , which is evenly divisible by 6.

INDUCTION ( $H[k] \Rightarrow H[k + 1]$ ). Assume that  $k^3 - k$  is evenly divisible by 6.  
Now,

$$\begin{aligned}(k + 1)^3 - (k + 1) &= (k^3 + 3k^2 + 3k + 1) - (k + 1) && \text{Expanding } (k + 1)^3 \\ &= (k^3 + 3k^2 + 2k) && \text{Combining like terms} \\ &= k(k^2 + 3k + 2) && \text{Distributing } k \\ &= k(k + 1)(k + 2) && \text{Factoring } k^2 + 3k + 2\end{aligned}$$

By Proposition A (below) this number is divisible by 3. And since at least one of  $k$ ,  $k + 1$ , or  $k + 2$  is an even number, it is also divisible by 2. That is,  $k(k + 1)(k + 2) = 2 \cdot 3 \cdot q$ , for some  $q$ , and is therefore divisible by 6.

**Proposition A.** For all  $n \in \mathbb{N}$ ,  $n(n + 1)(n + 2)$  is divisible by 3.

PROOF: by induction on  $n$  with hypothesis,  $H[k] \equiv k(k + 1)(k + 2) = 3p$  for some  $p$ .

BASE CASE:  $0 \cdot 1 \cdot 2 = 0 = 3 \cdot 0$ .

INDUCTION CASE: Assume  $k(k + 1)(k + 2) = 3p$  for some  $p$ .

$$\begin{aligned}(k + 1)(k + 2)(k + 3) &= k(k + 1)(k + 2) + 3(k + 1)(k + 2) && \text{(distributing over } k + 3) \\ &\stackrel{\text{IH}}{=} 3p + 3(k + 1)(k + 2) && \text{(Induction Hypothesis)} \\ &= 3[p + (k + 1)(k + 2)] && \text{(distributing the 3)}\end{aligned}$$

COMMENT: It is reasonable to say that this proposition is obvious and need not be proven.

4. Prove that for all natural numbers  $n > 4$ ,  $2^n > n^2$ .

SOLUTION

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The proof is by induction on  $k$  with hypothesis  $H(k) \equiv 2^k > k^2$ .

BASE CASE: To prove  $H(5)$ ,

$$2^5 = 32 > 25 = 5^2$$

INDUCTION STEP: Assume  $2^k > k^2$ . Now,

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2 \cdot k^2 && \text{(induction hypothesis, used twice)} \\ &= k^2 + k^2 \\ &> k^2 + 2k + 1 && (k > 4 \Rightarrow k^2 > 2k + 1) \\ &= (k + 1)^2 \end{aligned}$$

The induction is finished once it is established that for any natural number  $n > 4$ ,  $n^2 > 2n + 1$ . This fact is proved in the lemma below.

LEMMA For all  $n > 3$ ,  $n^2 > 2n + 1$ .

PROOF: The proof is by induction on  $k$  with hypothesis  $H(k) \equiv k > 3$ ,  $k^2 > 2k + 1$ .

BASE CASE:

$$3^2 = 9 > 7 = 3 \cdot 3 + 1$$

INDUCTION STEP: Assume  $k^2 > 2k + 1$ ,  $k > 3$ . Then

$$\begin{aligned} (k + 1)^2 &= k^2 + 2k + 1 \\ &> (2k + 1) + 2k + 1 && \text{(induction hypothesis)} \\ &> (2k + 1) + 1 + 1 && (k > 1) \\ &= (2k + 2) + 1 && \text{(arithmetic)} \\ &= 2(k + 1) + 1 && \text{(arithmetic)} \end{aligned}$$

This completes the induction and the proof of the lemma.  
The proof of the lemma completes the proof of the problem.

5. HINT: *Solving this problem requires more than a straightforward algebraic derivation. Think outside the box.*

Prove that for all natural numbers  $n$ ,  $\sum_{i=0}^n i^3 = \left(\sum_{i=0}^n i\right)^2$ .

SOLUTION

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SOLUTION NEEDED

6. Recall that the *choose* function, defined

$$\binom{s}{r} = \frac{s!}{r!(s-r)!}$$

is the number of different ways to choose  $r$  objects from a set of  $s$  objects. Prove that for all  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^{n-1} (2i)^2 = \binom{2n}{3}$$

SOLUTION

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*Solution needed.*

7. For  $n \in \mathbb{N}$ , let  $Q, P_1, P_2, \dots, P_n$  be propositions. Prove the following:

$$\begin{array}{ll} \text{(a)} & Q \wedge \bigvee_{i=1}^n P_i = \bigvee_{i=1}^n (Q \wedge P_i) \\ \text{(b)} & \neg \bigwedge_{i=1}^n P_i = \bigvee_{i=1}^n (\neg P_i) \end{array} \quad \begin{array}{ll} \text{(c)} & Q \vee \bigwedge_{i=1}^n P_i = \bigwedge_{i=1}^n (Q \vee P_i) \\ \text{(d)} & \neg \bigvee_{i=1}^n P_i = \bigwedge_{i=1}^n (\neg P_i) \end{array}$$

SOLUTION

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SOLUTION NEEDED

8. Consider the program

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 $\mathcal{P}$ : begin
  {A, B > 0}
  q := 0;
  r := A;
 $\ell$ : while r ≥ B do
  begin
    q := q + 1;
    r := r - B
  end;
end {A = qB + r ∧ r < B}

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Use Theorem 4.2 and invariant assertion

$$I \equiv A = qB + r$$

to prove that this program computes the quotient and remainder of A and B.

SOLUTION

According to Theorem 4.2, if we can show that assertion I is true when  $\mathcal{P}$  first reaches  $\ell$  and that executing the loop body leaves I true, then when  $\mathcal{P}$  ends, I will be true and the test “ $r \geq B$ ” will be false.

Thus, there are three things to prove: *Note: Blue comments need not be included because they are implied by the use of Theorem 4.2*

1. INITIALIZATION (BASE CASE):

$$\{A, B > 0\} \quad q := 0 \quad ; \quad r := A \quad \{A = qB + r\}$$

Since we have just assigned 0 to q and A to r we have

$$qB + r = 0 \cdot B + A = A.$$

2. INVARIANCE (INDUCTION CASE):

$$\{A = qB + r \wedge r \geq B\} \quad q := q + 1 \quad ; \quad r := r - B \quad \{A = qB + r\}$$

The loop body computes new values for program variables  $q' = q + 1$  and  $r' = r - B$  (valid because  $r \geq B$ ). So we have

$$\begin{aligned}
 q'B + r' &= (q + 1)B + (r - B) && \text{(substituting for } r' \text{ and } q') \\
 &= qB + B + r - B \\
 &= qb + r && \text{(simplifying)} \\
 &\stackrel{\text{IH}}{=} A && \text{(assumption)}
 \end{aligned}$$

as desired.

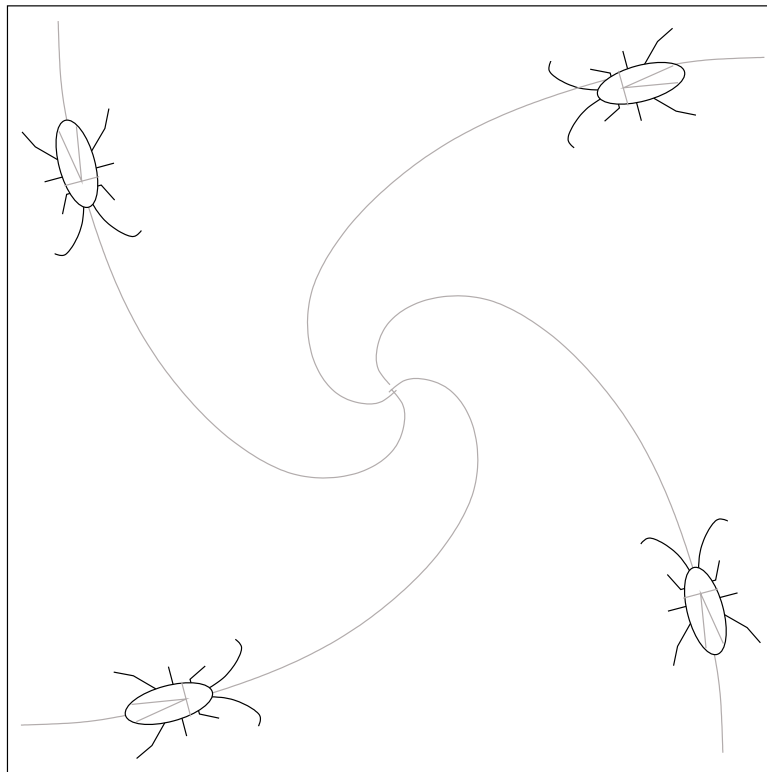
3. TERMINATION: It remains to show that  $I \wedge (r < b)$  implies the postcondition,  $\{(A = qB + r) \wedge (r < B)\}$ . But in this case the two formulas are identical, so the implication is tautologically valid.

9. SUPPLEMENTAL PROBLEM COMMENT: *This is neither a calculus question nor a logic puzzle.*

At last! You've gotten out of the dorms and into an apartment. Life can begin. You get moved in and a few days later are looking for a brownie pan. When you open the cabinet door, you see a 12-inch by 12-inch pan, just perfect for your cooking needs. That's the good news.

Unfortunately, there are four cockroaches in the pan, one at each corner. Now you may not have known this, but cockroaches are very amorous creatures, and will always move directly toward the object of their affection. Each of the four in your pan is attracted to the one in the adjacent corner, going counter-clockwise. So all four cockroaches simultaneously start walking toward the one that attracts them. As a result, they start spiraling toward the center of the pan (See the diagram below).

QUESTION: *How far will the cockroaches travel?*



## SOLUTION

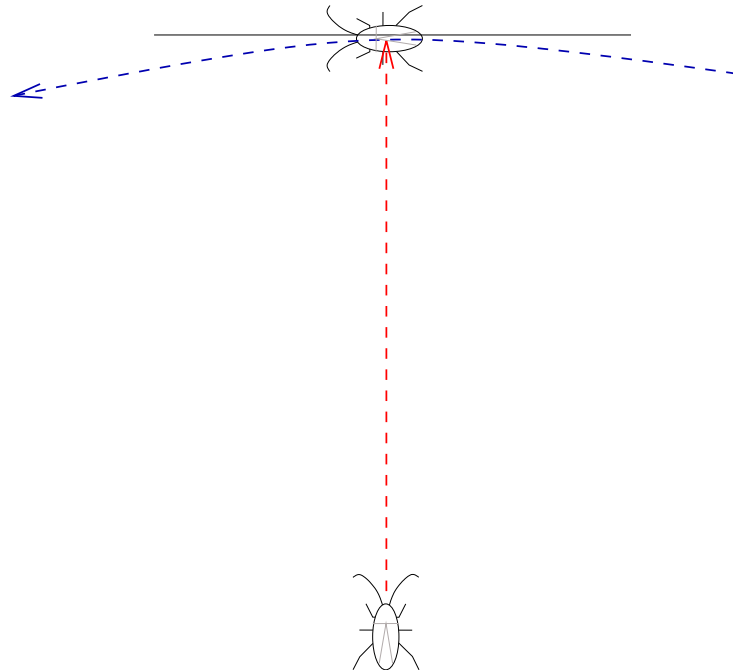
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Each cockroach travels the length of one side of the square pan, 12 inches in the problem as stated.

To gain the necessary insight, you have to look at the problem from the (any) cockroach's point of view. From that perspective,

- (a) the cockroach is moving directly toward its target, perpendicular to the (tangent) of the target's path of travel. We know from geometry that this is the shortest path it can take.
- (b) We also know (from common sense), that the total distance a cockroach travels is  $d = \alpha + \beta$ , where  $\alpha$  is the distance it has already traveled and  $\beta$  is how far it has yet to go. When the cockroach takes a step,  $\alpha$  and  $\beta$  change by the same distance  $s$ , but  $d = (\alpha + s) + (\beta - s)$  is unchanged.

These two properties are invariant—they do not change throughout cockroach's trip from the corner of the pan to its destination. From the cockroach's point of view, it is the pan that is turning, not itself. So by (a), above, it is invariantly traveling along the shortest path to its destination; and the total distance (b) is invariant also. Initially,  $d = 12$  inches, so that is how far it travels.



REMARK: *The COMMENT at the beginning of the problem says that you don't need calculus to solve the problem, but this is not entirely true. What you do need is a grasp of the underlying concepts of calculus. The solution is only valid in the limit. The true distance is some multiple of the number of discrete steps taken by the cockroach. Twelve inches is the limit of that number as the length of each stride becomes infinitesimal.*<sup>1</sup>

*One can reasonably argue (as I suspect your Calculus instructor would) that the "solution" given above needs to be verified by a rigorous mathematical formulation by which the limit value can be calculated. It is merely an idea about how to approach the proof. It is worthwhile to set up that formulation. On the other hand, if you were to write a program to approximate the limit, discovering the problem's invariants key to the computing the answer.*

END REMARK

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<sup>1</sup>Here's an interesting fact about cockroaches. Most insects have two forms of locomotion or *gaits*. One is *walking* in which two outer legs and to opposite middle leg work in unison. The three legs act as a tripod, and the two tripods alternate, propelling the the insect forward in a bi-pedal manner. The other gait is *metachronal*: the legs on either side thrust in sequence, resulting in a "wave" traversing from front to back, like the legs of a centipede or caterpillar. Cockroaches are unusual insects in that they have *three* gaits. In flight, a cockroach can literally *run* on its two hind legs, keeping the other four raised off the ground.