C241 Homework Assignment 6

NOTE: *If you cannot finish some of these problems, try at least to "set up" the answer, showing what you would need to do to complete the solution. It is important to demonstrate that you are following a valid approach.*

1. Let $a \in \mathbb{N}$ and define the set $\{t_0, t_1, \ldots, t_k, t_{k+1}, \ldots\}$ as follows:

$$\begin{aligned} t_0 &= a \\ &\vdots \\ t_{k+1} &= t_k + k + 1 \end{aligned}$$

(a) List the first ten values, t_0 through t_9

(b) Prove by induction on n: *For all* $n \in \mathbb{N}$, $t_n = a + \dfrac{n^2 + n}{2}$.

2. Let $a \in \mathbb{R}$ and define the set $\{t_0, t_1, t_2, \ldots, t_k, t_{k+1}, t_{k+2}, \ldots\}$ as follows:

$$\begin{aligned} t_0 &= 0 \\ t_1 &= \tfrac{1}{2} + a \\ &\vdots \\ t_{k+2} &= 2t_{k+1} - t_k + 1 \end{aligned}$$

(a) List the first ten values, t_0 through t_9

(b) Prove: *For all* $n \in \mathbb{N}$, $t_n = \dfrac{n^2}{2} + an.$

3. Use Theorem 4.2 to prove that the program below computes $A!$.

```
{x = A}
begin
z := 1;
while  x ≠ 1 do {z · x! = A!}
  begin
  z := z * x;
  x := x − 1
  end
end
{z = A!}
```

4. Definition. The *greatest common divisor* of two whole numbers n and m, written $\gcd(n, m)$, is the largest number that evenly divides n and m.

For example

$$\begin{aligned}
\gcd(24, 36) &= 12 \\
\gcd(55, 72) &= 1 \\
\gcd(55, 99) &= 11 \\
\gcd(1, k) &= \gcd(1, k) = 1 \text{ for any } k \\
\gcd(k, \ell) &= \gcd(\ell, k) \text{ for any } k, \ell \in \mathbb{W}
\end{aligned}$$

Use Theorem 4.2 to prove that the program below computes $\gcd(A, B)$.

```
{A,  B ∈ 𝕎}
begin
x  :=  A;
y  :=  B;
while  (x  ≠  y) do {gcd(x,  y)  =  gcd(A,  B)}
  begin
  if x  <  y
     then  y  :=  y − x
     else  x  :=  x − y
  end
end
{x  =  gcd(A,  B)}
```

More specifically,

(a) State the *three* things you must prove to verify that the program above computes the greatest common divisor.

(b) Prove these three things, if you can; or if not, reduce the arguments to *purely mathematical* propositions.

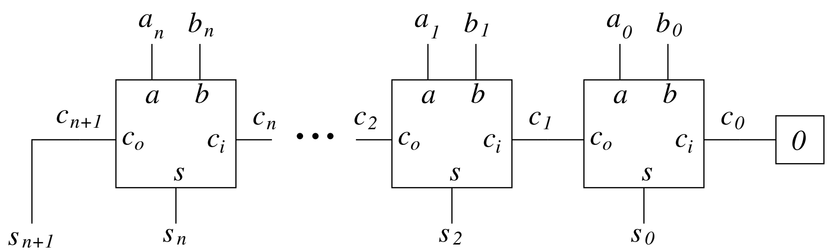
5. Prove: *If sets A and B are countable, then so are $A \cup B$, $A \times B$ and $A \setminus B$.*

6. Let A be a finite alphabet. Recall (Defn. 1.9) that A^* is the set of all words, including the empty word ε, over A.

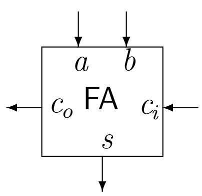
Is A^* countable? Explain your reasoning.

7. Supplemental Problem

In Section 2.5, the design is described of a digital circuit for binary addition. An *n-bit adder* is and array of n single-bit *full adders*



Each full adder, implements the truth table below, which reduces to the system of boolean equations on the right.



a	b	c_i	s	c_o
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

$$\begin{aligned} s &= a \oplus b \oplus c \\ c &= a\,b + a\,c + b\,c \end{aligned}$$

We would like to *prove* that this circuit actually adds. In order to do so, we must first define what *number* an n-bit binary *numeral* represents.

> **Definition.** *A binary numeral* $B = b_n\, b_{n-1} \cdots b_2\, b_1\, b_0$ *Represents the number*
>
> $$\mathcal{N}[B] \stackrel{\text{def}}{=} \sum_{i=0}^{n} \hat{b_i} \cdot 2^i \;\; \textit{where}\;\; \hat{b} = \begin{cases} \textit{the} \text{ number } 0 \textit{ if } b \textit{ is digit } 0 \\ \textit{the} \text{ number } 1 \textit{ if } b \textit{ is digit } 1 \end{cases}$$

Thus, for example,

$$\mathcal{N}[1011] = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 1 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 1 \cdot 1 = 11$$

Using this notation, the property we want to prove is:

> **Theorem.** *Let* $A \equiv a_{n-1} \cdots a_1\, a_0$ *and* $B \equiv b_{n-1} \cdots b_1\, b_0$ *be two n-bit binary numerals, and let* $S \equiv s_n \cdots s_1\, s_0$ *be the* $(n+1)$*-bit numeral generated by an n-bit binary adder with inputs A, B, and* $c_0 = 0$*. Then*
>
> $$\mathcal{N}[S] = \mathcal{N}[A] + \mathcal{N}[B]$$

Prove this Theorem.