# C241 Fall 2009
# Midterm II: Sets, Functions, Relations, General Induction

## Sets:

**1) Match each statement regarding sets with its equivalent logical statement**

| | | |
|---|---|---|
| ____ | i) $x \in (A \cup B)$ | a) $\neg \exists x[(x \in A)]$ |
| ____ | ii) $x \in (A \cap B)$ | b) $[(x \in A) \vee (x \in B)]$ |
| ____ | iii) $A \subseteq B$ | c) $\forall x[(x \in A) \rightarrow (x \in B)]$ |
| ____ | iv) $A = \emptyset$ | d) $[(x \in A) \wedge (x \in B)]$ |

**2) Label each of the following statements as either True or False. If a statement is False, correct it.**

$A = \{a, b, c, d\} \quad B = \{b, c, e\}$

a) $a \in A \cup B$

b) $d \in A \cap B$

c) $\{b, d\} \in A$

d) $(b, b) \in A \times B$

e) $\emptyset \in A$

f) $\{c, e\} \in$ the power set of $B$

## Functions:

**3) Match each of the functions below with its appropriate label (*you'll use each label exactly once)*. Under each function, *very briefly* justify your choice of label in plain English.**

__Labels__

i) Not a Function
ii) Function, but not injective nor surjective
iii) Injective Function but not Surjective
iv) Surjective Function but not Injective
v) Bijection

__Functions__

a) $A = \{a, b, c\}$ $B = \{1, 2, 3, 4\}$
$F : A \to B$ $F = \{(a, 1), (b, 1), (c, 2)\}$
Label:_____
Why?

b) $A = \{a, b, c\}$ $B = \{1, 2, 3, 4\}$
$F : A \to B$ $F = \{(a, 2), (b, 3), (a, 4)\}$
Label:_____
Why?

c) $A = \{a, b, c\}$ $C = \{y, z\}$
$F : A \to C$ $F = \{(a, y), (b, y), (c, z)\}$
Label:_____
Why?

d) $F : \mathbb{N} \to \mathbb{N}$ $F(x) = x^2$
Label:_____
Why?

e) $\mathbb{Z}$ =the integers. $F : \mathbb{Z} \to \mathbb{Z}$ $F(x) = x + 1$
Label:_____
Why?

**4)Prove that if $|A| \geq |B|$, then it's possible to define a surjective function $F : A \to B$**

## Relations: Graphs, Equivalence Relations, Partial Orders:

**5) Given that** $A = \{a, b, c, d\}$**,** $R \subseteq A \times A$**, and** $R = \{(a, a), (a, b), (b, a), (b, b), (a, d), (b, d), (c, c), (d, d)\}$**. First, draw the graph for** $R$**. Then list which of the following properties it has: Reflexive, Symmetric, Anti-symmetric, Transitive.**

**6)Prove that the following is an equivalence relation (write a clear explanation, don't just draw a graph).** $A =$ **a bag of M&M Candies (6 red, 6 yellow, 4 green, 7 blue, 3 orange).** $R \subseteq A \times A$**,** $R = \{(x, y) \mid$ **x is the same color as y**$\}$ .

**7)Prove that the following is a Partial Order. Then, decide whether or not it is a Total Order and prove that your decision is correct.** $A = \{\{a, b\}, \{c, d, e\}, \{f\}, \{g, h, i, j\}\}$**.** $R \subset A \times A$**.** $R = \{(B, C) \mid$ **it's possible to create a surjection** $F : B \rightarrow C$ $\}$ .

## Numerical Induction:

**8) Use numerical induction to prove:** $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

## Structural Induction

**9) Below is the inductive definition for the language $L$ and the definition of the recursive function $C : L \rightarrow N$. Define a new recursive function $D : L \rightarrow L$ which doubles each occurrence of $b$ in a string (so $D(baba) = bbabba$).**

| | $L(\{a, b\}^*)$ |
|---|---|
| 1. | $\epsilon \in L$ |
| 2a. | $s \in L \Rightarrow \text{a}s \in L$ |
| 2b. | $s \in L \Rightarrow \text{b}s \in L$ |

| | $C : L \rightarrow \mathbb{N}$ |
|---|---|
| 1. | $C(\epsilon) = 0$ |
| 2a. | $C(\text{a}s) = 1 + C(s)$ |
| 2b. | $C(\text{b}s) = C(s)$ |

**10)Use induction to prove that $\forall s \in L$, $C(D(s)) = C(s)$**

**11)Define two recursive functions on the of set binary trees (the set $T$): The height function $h(T) : T \rightarrow \mathbb{N}$ (where the height of a binary tree is the length of the longest path from the root node to a leaf), and the size function $s(T) : T \rightarrow \mathbb{N}$ (where the size of a binary tree is the total number of nodes in the tree, including leaves). Note that $h(\circ) = 0$ and $s(\circ) = 1$. Then use induction to prove that $\forall t \in T$, $2^{h(t)+1} \geq s(t)$.**

**Bonus: (extra credit)**

**Give an inductive definition for the set of all strings with an equal number of $a$'s and $b$'s (this is tricky). Then explain why you *can't* use the induction techniques we've learned so far (for example, induction on the length of the string) to prove that *all* strings with an equal number of $a$'s and $b$'s are included in your set (note that this is different than proving that all strings in your set have an equal number of $a$'s and $b$'s).**