C241 Fall 2009
Practice Exam

Important Note: This test is good practice for the final exam, but it is not, itself, the final exam. Topics we've covered in class (which might be on the real final but are not on this one) include, for example: formal logic proofs, set operations including the empty-set, proofs about properties of functions, equivalence relations, numerical induction on things other than summations, and tree traversals.

Answers will be posted later this week, but solutions to similar problems (or sometimes, the *same* problems) can be found in the solutions for previous assignments.

Logic (15 points)

1) Answer each of the following questions with "yes", "no", or "maybe". The predicates involved are: $C(x) =$"x has chocolate chips", $N(x) =$"x has walnuts", $E(z, x) =$"z ate x". The domain of x is cookies, and the domain of z is C241 students.

If it's true that $\forall x : C(x)$

a) Is there any cookie which is not chocolate chip?

b) Does any cookie have nuts?

If it's true that $\forall x, [\exists z : E(z, x)]$

c) Were all of the cookies eaten?

d) Did one person eat all the cookies?

If it's true that $\exists z, [\forall x : E(z, x)]$

e) Did one person eat all the cookies?

If it's true that $\forall x : [C(x) \leftrightarrow (\exists z : E(z, x))]$

f) Were all of the chocolate chip cookies eaten?

g) If a cookie didn't have chocolate chips, did anyone eat it?

2) Simplify, if possible, the following logical statement. (A list of the laws of logic is attached to the back of this exam):

$(w \wedge z \wedge x) \vee (w \wedge z \wedge \neg x) \vee (w \wedge \neg z \wedge x)$

Combinatorics (10 points)

2) As I'm creating an undirected graph with the vertices: $\{a, b, c, d, e, f\}$, I can pick any two different vertices and draw an edge between them (since it's an undirected graph, it won't matter which vertex I pick first). How many total different edges can I make in this graph? Explain briefly how you get your answer.

3) Idiot Sort is a very inefficient sorting algorithm. When given a set of n numbers, it just checks every possible arrangement of numbers until it encounters the arrangement that's in sorted order. In the worst case, how long will Idiot Sort take to sort the list?

Sets, Functions, and Relations (15 points)

4) Say that P is the set of all prime integers, L is the set of integers less than 10, and T is the multiples of 3. Then list the elements of the set:

$(\mathbb{N} \cap L) \cap \overline{(P \cup T)}$.

5) If $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, give an example of a function $F : A \to B$ which is neither injective nor surjective.

6) If $A = \{a, b, c, d\}$, give an example of a relation $R \subseteq A \times A$ which is a partial ordering. Is your relation also a total ordering (justify your answer)?.

Induction (20 points)

7) Use numerical induction to prove: $\sum_{i=1}^{n}(2i-1)=n^2$

8) Use induction to prove that for all $w \in L$, $C(w)$ is an even number.

	L
1.	$\bullet \in L$
2a.	$u \in L \Rightarrow \lhd u \rhd \in L$
2b.	$u, v \in L \Rightarrow \lhd u \bowtie v \rhd \in L$

	$C : L \rightarrow \mathbb{N}$
1.	$C(\bullet) = 0$
2a.	$C(\lhd u \rhd) = 2 + C(u)$
2b.	$C(\lhd u \bowtie v \rhd) = 2 + C(u) + C(v)$

Time Complexity (20 points)

9) Find the asymptotic complexity (the closest fitting Big-O: $O(1)$, $O(log_2(n))$, $O(n\ log_2(n))$, $O(n^2)$, **etc...) for the following (you don't need to justify your answer):**

a) $n^3 + 3n^2$

b) $2^n + n + 6$

c) $lg(n) + \frac{1}{n}$

d)

```
for(i : 1 to n)
   x = x + 1
```

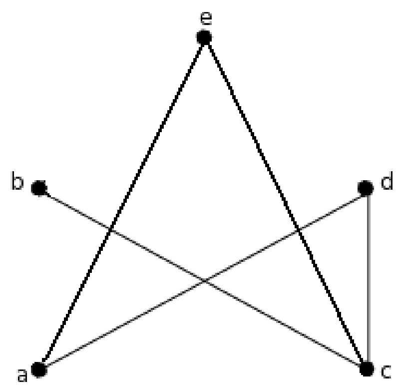
e)

```
for(i : 1 to n)
   for (j:1 to i)
      x = x + 1
```

10)Use the formal definition of Big-O to prove that: $5n^2 + n \in O(n^2)$

10)Use the formal definition of Big-O (and a proof by contradiction) to prove that: $n^2 \notin O(2n)$

Graphs (20 points)



11) Given a graph: $G = \{V, E\}$, an 'Independent Set' is a subset of vertices $V_I \subseteq V$ such that $[v_1, v_2 \in V_I] \Rightarrow \neg[\exists(v_1, v_2) \in E]$. In other words, V_I is a set of vertices which are *not* neighbors in G.

a) List all the vertices that are neighbors of e in the graph above.

b) Give an example of an independent set of size 2 in the graph above. Give an example of an independent set of size 3.

c) Draw a graph with 4 vertices that has an independent set of size 4. Draw a graph with 4 vertices whose largest independent set has size 1.

d) Give an example of a spanning tree for the graph above.

Bonus) The inverse of a graph $G = (V, E)$ is the graph $G' = (V, E')$, where $E' = (V \times V) \cap \overline{E}$. Draw the inverse for the graph above. Prove that if V_I is an independent set in G, then V_I will be a clique in G.

12) On the next page is the pseudo-code for Depth First Search on graphs, as well as an example graph. Use them to answer the following questions.

a)List out the vertices in the order they would be reached in a depth first search.

b)Explain precisely in clear, detailed english how you could adapt the Depth First Search algorithm to return the number of vertices that are *reachable* from vertex a.

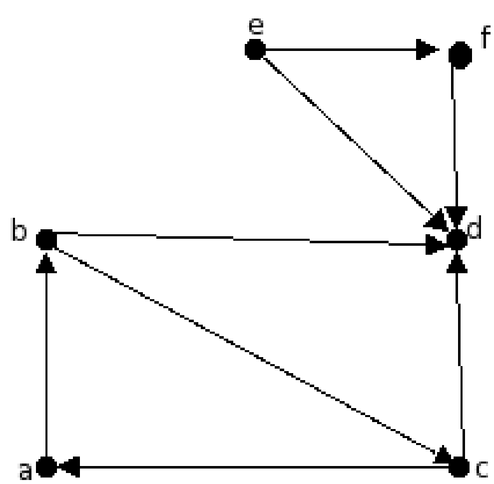
```
Outer_DFS(v)
{
    initialize_visited();    //the visited array keeps track of the
                             //vertices we've visited

    for each v in V;         //V is the set of all vertices in the graph
   {
          if not( visited[v] == 1 )
          { Inner_DFS(v); }
    }
}



Inner_DFS(v)
{
    visited[v] = 1;

    for each n in neighbors(v);//the neighbors of v are the vertices that
    {                          //have an edge leading to them from v
         if not(visited[n] == 1)
         { Inner_DFS(n);}
     }
 }
```



A Chart of the Algebraic Laws of Logic

(An extension of the chart on page 75 in your textbook)

Note: The symbols P,Q,R,K below can represent complex logical statements.

Double Negation	$\neg(\neg P)$	$\equiv$	P		
Identity Laws	$(P \vee F)$	$\equiv$	P		
	$(P \wedge T)$	$\equiv$	P		
	$(P \vee T)$	$\equiv$	T		
	$(P \wedge F)$	$\equiv$	F		
Complement Laws	$(P \vee \neg P)$	$\equiv$	T		
	$(P \wedge \neg P)$	$\equiv$	F		
Idempotence Laws	$(P \vee P)$	$\equiv$	P		
	$(P \wedge P)$	$\equiv$	P		
Commutative Laws	$(P \vee Q)$	$\equiv$	$(Q \vee P)$		
	$(P \wedge Q)$	$\equiv$	$(Q \wedge P)$		
Associative Laws	$(P \vee (Q \vee R))$	$\equiv$	$((P \vee Q) \vee R)$	$\equiv$	$(P \vee Q \vee R)$
	$(P \wedge (Q \wedge R))$	$\equiv$	$((P \wedge Q) \wedge R)$	$\equiv$	$(P \wedge Q \wedge R)$
DeMorgan's Laws	$\neg(P \vee Q)$	$\equiv$	$(\neg P \wedge \neg Q)$		
	$\neg(P \wedge Q)$	$\equiv$	$(\neg P \vee \neg Q)$		
Distributive Laws	$P \vee (K \wedge Q)$	$\equiv$	$(P \vee K) \wedge (P \vee Q)$		
	$P \wedge (K \vee Q)$	$\equiv$	$(P \wedge K) \vee (P \wedge Q)$		
Subsumption Laws	$P \vee (P \wedge Q)$	$\equiv$	P		
	$P \wedge (P \vee Q)$	$\equiv$	P		
Definition of Implication	$(P \rightarrow Q)$	$\equiv$	$(\neg P \vee Q)$		