# C241 Homework 1: Inductive Set Definitions and Basic Induction

# Due Wednesday, 10/17/07

**1) Give an inductive definition for the following sets:**

i) The set of all strings of a's, b's and c's (for example: a, b, c, abcabb, ac, bbb...)

ii) The natural numbers (for example: 0, 1, 2, 3, 4, 5...)

iii) The set of all binary trees

iv) The set of all even natural numbers (for example: 0, 2, 4, 6...)

v) The set of all odd natural numbers (for example: 1, 3, 5, 7...)

vi) The set of all powers of two (for example: 1, 2, 4, 8, 16....)

vii) The set of even length strings of a's and b's (for example: aa, ba, ab, bb, aabb, babb, abba, bababaab... )

**2) Answer the following questions about the three sets defined below.**

| **Parens** | |
|---|---|
| Base Set: | $() \in Parens$ |
| Constructors: | 1. If $u \in Parens$ then $(u) \in Parens$ |
| | 2. If $u, v \in Parens$ then $(uv) \in Parens$ |

| **Affirm** | |
|---|---|
| Base Set: | $Yep, Yes, Woot \in Affirm$ |
| Constructors: | 1. If $u \in Affirm$ then $u! \in Affirm$ |
| | 2. If $u, v \in Affirm$ then $u \textit{ and } v \in Affirm$ |

| **Twos** | |
|---|---|
| Base Set: | $2 \in Twos$ |
| Constructor: | If $u, v \in Twos$ then $u + v \in Twos$ |

**a) Which of these strings are in the set Parens?**

| | | | |
|---|---|---|---|
| (i) | ( ( ( ) ) ) | (iv) | ( ( )( ) ) |
| (ii) | ( ) ( ) | (v) | ( ( ) ( ) ( ) ) |
| (iii) | ( ( ( ) ( ) ) ( ) ) | (vi) | ( ( ( ) ( ) ) ) |

**b) Which of these strings are in the set Affirm?**

| | | | |
|---|---|---|---|
| (i) | Yep | (iv) | Yep and ! |
| (ii) | Yes and Woot! | (v) | Yes and Yep and Woot |
| (iii) | Yes!!! | (vi) | Yes! Woot! |

**c) List five different numbers that are in the set Twos. Are all the numbers in the set Twos even? (explain your reasoning)**

**d) We say a definition is ambiguous if there's any item in the set that can be built in two different ways (using two different sequences of constructors). Show that Affirm and Twos are both ambiguous (give an example of an item from each set that can be built in two different ways).)**

**2) Use induction and your definition of the natural numbers from question 1 to prove that all natural numbers are positive.**

**3) Use induction to prove that every string in Twos is even.**

**4) Use induction to prove that all the strings in Parens have an equal number of left and right parentheses.**

**5) Use induction to prove that all binary trees have one more leaf than interior nodes**

Proofs:

**6) Easy Proof: Prove that the string (()()) is in the set Parens.**

**7) Medium Proof: Give a convincing explanation in plain english that *all* even numbers are in the set Twos.**

**8) (*Extra Credit*) Hard Proof: It turns out, weirdly, that the sum of the first $n$ odd numbers always equals $n^2$. Try it: the sum of the first two odd numbers = 1 + 3 = 4, which is $2^2$. The sum of the first four odd numbers = 1 + 3 + 5 + 7 = 16, which is $4^2$. Can you use induction on $n$ to prove that this always works? (Remember $n$ is a natural number, so you're proving a property of the natural numbers. That means your base case will be $n = 1$, and your induction step will be from $n$ to $n + 1$. It may also help to notice that $n^{th}$ odd number is equal to $2n - 1$)**