C241 Homework 1: Inductive Set Definitions and Basic Induction

Due Wednesday, 10/17/07

1) Give an inductive definition for the following sets:

i) The set of all strings of a's, b's and c's (for example: a, b, c, abcabb, ac, bbb...)

ABC

Base Set:	$a, b, c \in ABC$
Constructors:	1. If $S \in ABC$ then $aS \in ABC$
	2. If $S \in ABC$ then $bS \in ABC$
	3. If $S \in ABC$ then $cS \in ABC$

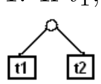
ii) The natural numbers (for example: 0, 1, 2, 3, 4, 5...)

$\mathbb{N}$

Base Set:	$0 \in \mathbb{N}$
Constructor:	1. If $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$

iii) The set of all binary trees

BinaryTree

Base Set:	$\circ \in BinaryTree$
Constructor:	1. If $t_1, t_2 \in BinaryTree$ then
	 $\in BinaryTree$

iv) The set of all even natural numbers (for example: 0, 2, 4, 6...)

EVEN

Base Set:	$0 \in EVEN$
Constructor:	1. If $n \in EVEN$ then $n + 2 \in EVEN$

v) The set of all odd natural numbers (for example: 1, 3, 5, 7...)

ODD

Base Set:	$1 \in ODD$
Constructor:	1. If $n \in ODD$ then $n+2 \in ODD$

vi) The set of all powers of two (for example: 1, 2, 4, 8, 16....)

POW2

Base Set:	$1 \in POW2$
Constructor:	1. If $n \in POW2$ then $2n \in POW2$

vii) The set of even length strings of a's and b's (for example: aa, ba, ab, bb, aabb, babb, abba, babab aab...)

EvenAB

Base Set:	$aa, bb, ab, ba \in EvenAB$
Constructors:	1. If $S \in EvenAB$ then $aaS \in EvenAB$
	2. If $S \in EvenAB$ then $bbS \in EvenAB$
	3. If $S \in EvenAB$ then $abS \in EvenAB$
	4. If $S \in EvenAB$ then $baS \in EvenAB$

2) Answer the following questions about the three sets defined below.

Parens

Base Set:	$() \in Parens$
Constructors:	1. If $u \in Parens$ then $(u) \in Parens$
	2. If $u, v \in Parens$ then $(uv) \in Parens$

Affirm

Base Set:	$Yep, Yes, Woot \in Affirm$
Constructors:	1. If $u \in Affirm$ then $u! \in Affirm$
	2. If $u, v \in Affirm$ then $u\ and\ v \in Affirm$

Twos

Base Set:	$2 \in Twos$
Constructor:	If $u, v \in Twos$ then $u + v \in Twos$

a) Which of these strings are in the set Parens?

(i) YES	((( )))	(iv) YES	(( )( ))
(ii) NO	( ) ( )	(v) NO	(( ) ( ) ( ))
(iii) YES	((( ) ( )) ( ))	(vi) YES	((( ) ( )))

b) Which of these strings are in the set Affirm?

(i) YES	Yep	(iv) NO	Yep and !
(ii) YES	Yes and Woot!	(v) YES	Yes and Yep and Woot
(iii) YES	Yes!!!	(vi) NO	Yes! Woot!

c) List five different numbers that are in the set Twos. Are all the numbers in the set Twos even? (explain your reasoning)

$2 \in Twos$, so $2+2=4 \in Twos$, so $4+2=6 \in Twos$, so $6+6=12 \in Twos$, so $12+4=16 \in Twos$... (which gives us 2,4,6,12,16 all in Twos). Yep, they'll all be even. You start with the number 2, and then you add together items in the set to construct new items... so every number in the set is either 2 or something created by adding some number of 2's together. So everything will be even.

d) We say a definition is ambiguous if there's any item in the set that can be built in two different ways (using two different sequences of constructors). Show that Affirm and Twos are both ambiguous (give an example of an item from each set that can be built in two different ways).)

There's many right answers here. Below are a couple examples.

For Affirm: we can create the string "Yes and Yep and Woot" in two ways: $[[Yes_{BS}\ and\ Yep_{BS}]_{C2}\ and\ Woot_{BS}]_{C2}$ (applying constructor 2 first to get the string "Yes and Yep", and then again to get "Yes and Yep and Woot").
Or: $[Yes_{BS}\ and\ [Yep_{BS}\ and\ Woot_{BS}]_{C2}]_{C2}$ (applying constructor 2 first to get the string "Yep and Woot", and then again to get "Yes and Yep and Woot").

For Twos: We can create 8 in two ways: First, we can start with $2+2=4$, which tells us $4 \in Twos$. Given that, we can apply the constructor again and get $4+4=8$. Or, we can get there by starting with $2+2=4$ again, to get $4 \in Twos$, and then using the constructor twice more with $v=2$: $4+2=6$ (so $6 \in Twos$), and then $6+2=8$.

2) Use induction and your definition of the natural numbers from question 1 to prove that all natural numbers are positive.

Base Case 0: Technically 0 isn't considered positive. But it's not negative either, so for the sake of doing this problem, let's assume it counts. (If you said the proof was impossible because 0 isn't positive, you got full credit).

Induction Step: Show that, if n is positive (Induction Hypothesis), then $n+1$ is also positive. Well, $n+1$ is bigger than n, so this is obviously true

3) Use induction to prove that every string in Twos is even.

Base Case 2: 2 is an even number.

Induction Step: Show that, if both n and m are even (Induction Hypothesis), then $n+m$ is also even. Since the sum of two even numbers is always an even number, this is true.

4) Use induction to prove that all the strings in Parens have an equal number of left and right parentheses.

Let's use the variable L_s to mean the number of left parentheses in the string s, and R_s to mean the number of right parentheses in the string s.

Base Case (): Since $L_{()} = 1$ (it has one left parenthesis) and $R_{()} = 1$ (it has one right parenthesis), and $1 = 1$, the base case has an equal number of left and right parentheses.

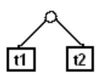
Induction Steps:
Show that, if u has an equal number of left and right parentheses (in other words, if $L_u = R_u$) (Induction Hypothesis),
then (u) has an equal number of left and right parentheses. Well $L_{(u)} = L_u + 1$ and $R_{(u)} = R_u + 1$. And, since $L_u = R_u$ (IH), then it's true that $L_u + 1 = R_u + 1$. (In other words, if u has an equal number of left and right parentheses, and we add one right and one left parenthesis, the result still has an equal number of left and right parentheses)

Show that, if u and v both have an equal number of left and right parentheses (in other words, if $L_u = R_u$ and $L_v = R_v$) (Induction Hypothesis),
then (uv) has an equal number of left and right parentheses. Well $L_{(uv)} = L_u + L_v + 1$ and $R_{(uv)} = R_u + L_v + 1$. And, since $L_u = R_u$ and $L_v = R_v$ (IH), then it's true that $L_u + L_v + 1 = R_u + R_v + 1$.

5) Use induction to prove that all binary trees have one more leaf than interior nodes

Base Case $\circ$: This has one leaf and no interior nodes, so it has one more leaf than interior nodes.

Induction Step: Let's say that t_1 has m leaves, and t_2 has n leaves. Show that if t_1 and t_2 both have one more leaf than interior nodes, (Induction Hypothesis), then



also has one more leaf than interior nodes. Well, this tree will have all the leaves from t_1 and t_2, so it'll have $m + n$ leaves. And it'll have all the interior nodes from t_1 and t_2 plus the new root we added. So (using our IH), this tree will have $(m-1) + (n-1) + 1 = m + n - 1$ interior nodes. Which means it'll have one more leaf than interior nodes.

Proofs:

6) Easy Proof: Prove that the string (()()) is in the set Parens.

Well, () is in Parens since it's in the base set. So if we use the second constructor, with $u = ()$ and $v = ()$, we get (()()). So (()()) is in Parens.

7) Medium Proof: Give a convincing explanation in plain english that *all* even numbers are in the set Twos.

Since 2 is in the base set, we can use the second constructor (with v set to 2) to add 2 to any number in our set, and the result will also be in our set. So, we get 2+2=4, 4+2=6, 6+2=8, 8+2=10... and so on. This will work just like our inductive definition for the even natural numbers. We'll be able to generate every even natural number.

8) (*Extra Credit*) Hard Proof: It turns out, weirdly, that the sum of the first n odd numbers always equals n^2. Try it: the sum of the first two odd numbers = 1 + 3 = 4, which is 2^2. The sum of the first four odd numbers = 1 + 3 + 5 + 7 = 16, which is 4^2. Can you use induction on n to prove that this always works? (Remember n is a natural number, so you're proving a property of the natural numbers. That means your base case will be $n = 1$, and your induction step will be from n to $n+1$. It may also help to notice that n^{th} odd number is equal to $2n - 1$)

Rather than writing out the sum as $1+3+5...+2n-1$, we're going to use the notation $\sum_{i=1}^{n}(2i-1)$, which means the same thing. Check out page 51 of your textbook for an explanation of this notation.

Base Case (n = 1):

$$\sum_{i=1}^{1}(2i-1) =^? 1^2$$

$$2(1) - 1 =^? 1$$

$$1 = 1$$

Induction Step: Assume$\sum_{i=1}^{n}(2i-1) = n^2$ (Induction Hypothesis).

Then show that this implies: $\sum_{i=1}^{(n+1)}(2i-1) = (n+1)^2$.

$$\sum_{i=1}^{(n+1)} (2i-1) =^? (n+1)^2$$

$$\sum_{i=1}^{n}(2i-1) + (2(n+1)-1) =^? (n+1)^2 \text{ (expand summation)}$$

$$n^2 + (2(n+1)-1) =^? (n+1)^2 \text{ (Induction Hypothesis)}$$

$$n^2 + (2n+1) =^? (n+1)^2 \text{ (basic algebra)}$$

$$n^2 + 2n + 1 = n^2 + 2n + 1 \text{ (multiply out the right side)}$$