C241 Homework 2: Induction Review, Basic Combinatorics

Due Wednesday, 10/24/07

1) Answer the following questions about the three sets defined below. These are additional practice with induction, and the solutions to the previous assignment may be a useful reference.

OddPal

Base Set:	$a, b, c \in OddPal$
Constructors:	1. If $S \in OddPal$ then $aSa \in OddPal$
	2. If $S \in OddPal$ then $bSb \in OddPal$
	3. If $S \in OddPal$ then $cSc \in OddPal$

EqualAB

Base Set:	$ab, ba \in EqualAb$
Constructors:	1. If $S \in EqualAB$ then $aSb \in EqualAB$
	2. If $S \in EqualAB$ then $bSa \in EqualAB$
	3. If $S \in EqualAB$ then $SS \in EqualAB$
	(so if $ba \in EqualAB$, then $baba \in EqualAB$)

1MoreA

Base Set:	$a \in 1MoreA$
Constructors:	1. If $S_1, S_2 \in 1MoreA$ then $bS_1S_2 \in 1MoreA$
	2. If $S_1, S_2 \in 1MoreA$ then $S_1bS_2 \in 1MoreA$
	3. If $S_1, S_2 \in 1MoreA$ then $S_1S_2b \in 1MoreA$

a) Prove that all strings in the set OddPal have odd length

b) Prove that all strings in the set OddPal are palindromes. A string is a palindrome if when you reverse it you get the same string back. So $abcba$ **is a palindrome (because its reverse is** $abcba$**). But** abc **is not a palindrome (because it's reverse is** cba**). For this proof, it's fine to use the fact that, if a string is a palindrome, you can add the same character to both ends (for instance if you add** a**'s to** bcb **you get** $abcba$**), and the resulting string will still be a palindrome.**

c) What could you add to the base set in OddPal so that it would also generate even length palindromes?

d) Prove that every string in EqualAB has an equal number of a's and b's

e) Prove that every string in 1MoreA has exactly one more a than b's

2) Match the cases below with the appropriate combinatorics formula.

a) $n!$

b) $\frac{n!}{k!(n-k)!}$

c) $m + n$

d) $\frac{n!}{(n-k)!}$

e) $\frac{(n+(p-1))!}{(p-1)!}$

f) $m \times n$

g) $m - x$

h) $\frac{n!}{k_1!k_2!}$

i) $\frac{(n+(p-1))!}{n!(p-1)!}$

1) How many ways can you choose one option from either a set of m options or a set of n options?

2) How many ways can you choose one option from a set of m options and then choose another option from a set of n options?

3) How many ways can you arrange n items in order (like arranging books in a line on a shelf)?

4) How many ways can you pick out k items from a group of n items (like picking books to put in a bag).

5) How many ways can you take k items from a set of n items, and arrange them in order?

6) How many visibly different ways can you arrange n items if k_1 of them are identical copies of one type of item, and k_2 of them are identical copes of another type of item?

7) How many ways can you arrange n items and split them into p groups (which requires p-1 partitions)? This is like arranging books across p shelves.

8) How many ways can you take n identical items and split them into p groups (which requires p-1 partitions)? This is like splitting n identical beers among p friends.

9) How many ways can you pick an option if there are m total options, but you want to avoid picking x of them?

3) You have a large collection of microbrew beer, one bottle from each of 99 different breweries. You also have two shelves on your wall which each hold 50 bottles, and you have a large trash can. Answer the following multiple choice questions.

a) How many ways can you arrange all 99 bottles in a line on the floor?

a) 99

b) 99!

c) 1

b) How many ways can you pick 5 bottles to throw out?

a) 5!

b) $\frac{99!}{(99-5)!}$

c) $\frac{99!}{5!(99-5)!}$

c) How many ways can you arrange 10 of the bottles on the top shelf?

a) $\frac{99!}{(99-10)!}$

b) 10!

c) $\frac{99!}{10!(99-10)!}$

d) Let's say 10 of your bottles are from Indiana, and 13 of them are from Ohio. If you're going to drink two bottles, one from Indiana and one from Ohio, how many ways can you choose these two bottles?

a) $10 + 13$

b) 10×13

e) Let's say 10 of your bottles are from Indiana, and 13 of them are from Ohio. If you're going to either drink a bottle from Indiana or Ohio to drink, how many ways can you choose this bottle?

a) $10 + 13$

b) 10×13

f) You thought you had 15 different bottles from Wisconsin. But really, it turns out three of them are the exact same type of beer from the same brewery. You can't tell these three bottles apart. How many distinguishably different ways can you arrange your Wisconsin bottles on the top shelf?

a) $\frac{15!}{3!}$

b) $\frac{15!}{3}$

c) $15! - 3!$

g) It's even worse with your Texas bottles. You thought you had 9 different bottles, but you really had three identical bottles of "rattler's brew", five identical bottles of "lone star lager", and one bottle of "tecate". How many distinguishably different ways can you arrange your Texas bottles on the top shelf?

a) $\frac{9!}{3!+5!}$

b) $\frac{9!}{3!\times 5!}$

h) You've got 10 beers from Ohio, and 8 different beers from Indiana. How many ways can you arrange all the Ohio beers on the top shelf and then all the Indiana beers on the second shelf?

a) $10! \times 5!$

b) $15!$

c) $10! + 5!$

i) You've got 12 beers from Louisiana. How many ways can you pick out 4 beers to give to your friend Jane and then 2 beers to give to your friend Jack?

a) $\frac{12!}{(12-4)!} \times \frac{12!}{(8-2)!}$

b) $\frac{12!}{4!(12-4)!} \times \frac{12!}{2!(8-2)!}$

c) $\frac{12!}{4!(12-4)!} \times \frac{8!}{2!(8-2)!}$

j) You actually have 5 very wide shelves to display your beer. How many ways can you arrange all 99 different bottles across these 5 shelves?

a) $\frac{(99+5)!}{99!\times 5!}$

b) $\frac{(99+5)!}{5!}$

c) $\frac{(99+4)!}{4!}$

k) It turns out you somehow got 8 identical bottles of "Lugubrious Lager". One bottle of that stuff is really more than you need. How many ways could you distribute those 8 bottles among 4 of your friends?

a) $\frac{8!}{4!}$

b) $\frac{(8+3)!}{3!}$

c) $\frac{(8+3)!}{3!8!}$

l) Let's say you have 50 different american bottles of beer. Specifically, you have 10 bottles from each of 5 different regions of the country (East Coast, South, Midwest, Southwest, and Plains States). You're having a party, and you want to let people sample some things from your collection. How many ways can you pick 2 bottles from each region of the country?

a) $5 \times \frac{10!}{2!(10-2)!}$

b) $(\frac{10!}{2!(10-2)!})^5$

m) Same situation as above, but it turns out only one guy showed up to your party. You'd like to pick just one region and give him two beers to try from it. How many ways can you do this?

a) $5 \times \frac{10!}{2!(10-2)!}$

b) $(\frac{10!}{2!(10-2)!})^5$

n) You have a small cooler for tailgating that will hold at most 5 bottles. Also, you have a superstition that taking an even number of bottles to the game would be bad luck. How many ways can you choose an odd number of your 99 bottles to put in the cooler?

a) $\frac{99!}{5!(99-5)!} \times \frac{99!}{3!(99-3)!} \times 99$

b) $\frac{99!}{5!(99-5)!} + \frac{99!}{3!(99-3)!} + 99$

4) Answer the following combinatorics problems.

a) Easter Eggs: You're painting Easter Eggs. If you paint 3 big stripes on each egg, and you have 5 colors to choose from (red, blue, yellow, green, purple), then how many visibly different eggs can you make?

b) Dinner: You're eating dinner at a dorm cafeteria. Tonight, there are 4 desert options, 5 entree options, and 6 side item options. **(i)** How many ways can you pick out two deserts? **(ii)** How many ways can you pick out 3 side items? **(iii)** If your meal plan covers 1 entree, 3 side items, and 2 deserts, how many different meals can you order?)

c) Applied Problem Sliding 8 Puzzle: A problem that's often taught in AI courses is the Sliding 8 puzzle. This is a 3x3 square containing 8 sliding tiles (with the 9th spot is left open). The tiles are numbered 1 through 8. If a tile is next to the open spot, it can be slid into the open spot, which leaves its old spot open for another tile to slide into, and so on. The goal of the puzzle is to start with the tiles randomly arranged in the square, and then have your AI program find a way to slide the tiles around the square and back into 1-8 order. In order to do this, the program searches through the different possible configurations of the board. How many ways can you arrange 8 different tiles and a blank spot across 9 possible spaces?

d) Coin Flip Problem: **(i)** If you flip a coin 5 times, how many different sequences of Heads and Tails can you get? Think of a given sequence as being a string of H and T symbols (so HTTTT represents the case where only the first flip comes up Heads). **(ii)** How many ways can you get 3 Heads? **iii** How many total ways can you get more Heads than Tails? Go ahead and evaluate the factorials and exponents to find the actual numerical answers for iii and i. **(iv)** How do the answers compare? Why does this make sense?)

Proofs:

6) Easy Proof: Prove that so long as k and n are both at least 2, there are always more ways to permute k of n items than there are to choose k of n items. (You don't need to use induction here or anything. Just look at the formulas. Then explain clearly and precisely why this is true).

7) Medium Proof: Prove that the number of ways you can choose a group of k items from a set of n items, is the same as the number of ways you can choose a group of (n-k) items from a set of n items. (In other words, if you've got n books, picking k of them to take with you is basically the same as choosing (n-k) of them to leave behind.) Use the formulas to clearly explain why these two things are always equivalent.

8) (*Extra Credit*) Hard Proof: Problem 5.28 in your book gives an inductive proof of the binomial theorem. To determine the number of subsets of size k that can be picked from a set of n items, we use the choose formula. If we wanted to know the number of subsets that were of size k or of size j, we could use the choose formula for both and then sum the results. So, what would it look like if we were to use the choose formula to find the total number of subsets of *any* size from a set of n items? We've also discussed a better method of viewing this problem which produces the answer 2^n (what method was this?). This leaves us with two very different looking ways of finding the same value. Use the theorem proved in 5.28 to justify why these two methods produce the same number.