C241 Homework 2: Induction Review, Basic Combinatorics

Due Wednesday, 9/17/09

1) Answer the following questions about the three sets defined below. These are additional practice with induction, and the solutions to the previous assignment may be a useful reference.

OddPal

Base Set:	$a, b, c \in OddPal$
Constructors:	1. If $S \in OddPal$ then $aSa \in OddPal$
	2. If $S \in OddPal$ then $bSb \in OddPal$
	3. If $S \in OddPal$ then $cSc \in OddPal$

EqualAB

Base Set:	$ab, ba \in EqualAb$
Constructors:	1. If $S \in EqualAB$ then $aSb \in EqualAB$
	2. If $S \in EqualAB$ then $bSa \in EqualAB$
	3. If $S \in EqualAB$ then $SS \in EqualAB$
	(so if $ba \in EqualAB$, then $baba \in EqualAB$)

1MoreA

Base Set:	$a \in 1MoreA$
Constructors:	1. If $S_1, S_2 \in 1MoreA$ then $bS_1S_2 \in 1MoreA$
	2. If $S_1, S_2 \in 1MoreA$ then $S_1bS_2 \in 1MoreA$
	3. If $S_1, S_2 \in 1MoreA$ then $S_1S_2b \in 1MoreA$

a) Prove that all strings in the set OddPal have odd length

Base Case a, b, c: The strings a, b, c all have length 1, which is odd.

Induction Step: Show that, if string S has odd length (Induction Hypothesis), then aSa, bSb, and cSc also all have odd length. All three constructors add two characters to the length of S, and if the length of S is odd, then so is the length of S+2. So all three strings aSa, bSb and cSc will have odd length.

b) Prove that all strings in the set OddPal are palindromes. A string is a palindrome if when you reverse it you get the same string back.

So $abcba$ is a palindrome (because its reverse is $abcba$). But abc is not a palindrome (because it's reverse is cba). For this proof, it's fine to use the fact that, if a string is a palindrome, you can add the same character to both ends (for instance if you add a's to bcb you get $abcba$), and the resulting string will still be a palindrome.

Base Case a, b, c: The strings a, b, c are all clearly the same read backwards or forwards, so they're all palindromes.

Induction Step: Show that, if string S is a palindrome (Induction Hypothesis), then aSa, bSb, and cSc also all be palindromes. All three constructors add the same character to both ends of S. So, as we said above, if S is a palindrome then all three strings aSa, bSb and cSc will also be palindromes.

c) What could you add to the base set in OddPal so that it would also generate even length palindromes?

Technically we could add any even-length palindrome (such as abba), and this would make it possible to generate *some* even length palindromes (such as cab-bac). But if we add the strings aa, bb, cc to the base set, we'll be able to generate all even length palindromes that use the letters a, b, c.

d) Prove that every string in EqualAB has an equal number of a's and b's

Base Case ab, ba: Both of the strings ab, ba have one b and one a, so they clearly have an equal number of a's and b's.

Induction Step 1: Show that, if string S has an equal number of a's and b's (Induction Hypothesis),
then aSb and bSa also have an equal number of a's and b's. Well, both of these constructors add one a and one b to S. So if S has an equal number of a's and b's, then the strings aSb and bSa will also have an equal number of a's and b's.
Induction Step 2: Show that, if string S has an equal number of a's and b's (Induction Hypothesis),
then SS will also have an equal number of a's and b's. Let's say S has m a's and m b's (since we've assumed it has an equal number of a's and b's). Then SS (which takes the string S and writes it twice) has $2m$ a's and $2m$ b's, so it will also have an equal number of a's and b's.

e) Prove that every string in 1MoreA has exactly one more a than b's

Base Case a: The string a has 1 a and 0 b's, so it clearly has one more a than b's.

Induction Step 1: Show that, if strings S_1 and S_2 both have one more a than b's (Induction Hypothesis),
then $bS_1S_2, S_1bS_2, S_1S_2b$ all have one more a than b's. Let's say S_1 has m a's and thus (by our induction hypothesis) $m - 1$ b's. And let's say S_2 has n a's and thus $n - 1$ b's. In all three of these constructors we combined S_1, S_2 and b. Together, S_1 and S_2 have $m + n$ a's, and $m - 1 + n - 1 = m + n - 2$ b's. Then when we add a b to these strings, we get a string with $m + n$ a's and $m + n - 2 + 1 = m + n - 1$ b's. So each of these constructors will produce a string with exactly one more a than b's.

2) Match the cases below with the appropriate combinatorics formula.

1) How many ways can you choose one option from either a set of m options or a set of n options? c) $m + n$

2) How many ways can you choose one option from a set of m options and then choose another option from a set of n options? f) $m \times n$

3) How many ways can you arrange n items in order (like arranging books in a line on a shelf)? a) $n!$

4) How many ways can you pick out k items from a group of n items (like picking books to put in a bag). b) $\frac{n!}{k!(n-k)!}$

5) How many ways can you take k items from a set of n items, and arrange them in order? d) $\frac{n!}{(n-k)!}$

6) How many visibly different ways can you arrange n items if k_1 of them are identical copies of one type of item, and k_2 of them are identical copes of another type of item? h) $\frac{n!}{k_1!k_2!}$

7) How many ways can you arrange n items and split them into p groups (which requires p-1 partitions)? This is like arranging books across p shelves. e) $\frac{(n+(p-1))!}{(p-1)!}$

8) How many ways can you take n identical items and split them into p groups (which requires p-1 partitions)? This is like splitting n identical beers among p friends. f) i) $\frac{(n+(p-1))!}{n!(p-1)!}$

9) How many ways can you pick an option if there are m total options, but you want to avoid picking x of them? g) $m - x$

3) You have a large collection of microbrew beer, one bottle from each of 99 different breweries. You also have two shelves on your wall which each hold 50 bottles, and you have a large trash can. Answer the following multiple choice questions.

a) How many ways can you arrange all 99 bottles in a line on the floor?

a) 99

b) 99! Correct. [ways to order 99 items]

c) 1

b) How many ways can you pick 5 bottles to throw out?

a) 5!

b) $\frac{99!}{(99-5)!}$

c) $\frac{99!}{5!(99-5)!}$ Correct. [ways to choose unordered group of 5 items from 99]

c) How many ways can you arrange 10 of the bottles on the top shelf?

a) $\frac{99!}{(99-10)!}$ Correct. [ways to arrange 10 of 99 items in order]

b) 10!

c) $\frac{99!}{10!(99-10)!}$

d) Let's say 10 of your bottles are from Indiana, and 13 of them are from Ohio. If you're going to drink two bottles, one from Indiana and one from Ohio, how many ways can you choose these two bottles?

a) $10 + 13$

b) 10×13 Correct. [ways to make pick 1 of 10 options, and then pick 1 of 13]

e) Let's say 10 of your bottles are from Indiana, and 13 of them are from Ohio. If you're going to either drink a bottle from Indiana or Ohio to drink, how many ways can you choose this bottle?

a) $10 + 13$ Correct. [ways to pick 1 of 10 options or 1 of 13 options]

b) 10×13

f) You thought you had 15 different bottles from Wisconsin. But really, it turns out three of them are the exact same type of beer from the same brewery. You can't tell these three bottles apart. How many distinguishably different ways can you arrange your Wisconsin bottles on the top shelf?

a) $\frac{15!}{3!}$ Correct. [ways to arrange 15 items if 3 are identical]

b) $\frac{15!}{3}$

c) $15! - 3!$

g) It's even worse with your Texas bottles. You thought you had 9 different bottles, but you really had three identical bottles of "rattler's brew", five identical bottles of "lone star lager", and one bottle of "tecate". How many distinguishably different ways can you arrange your Texas bottles on the top shelf?

a) $\frac{9!}{3!+5!}$

b) $\frac{9!}{3!\times 5!}$ Correct. [ways to arrange 15 items including 3 identical copies of one item, and 5 identical copies of another]

h) You've got 10 beers from Ohio, and 8 different beers from Indiana. How many ways can you arrange all the Ohio beers on the top shelf and then all the Indiana beers on the second shelf?

a) $10! \times 5!$ Correct. [ways to order 10 items (on top shelf)] $\times$ [ways to order 5 items (on bottom shelf)].

b) $15!$

c) $10! + 5!$

i) You've got 12 beers from Louisiana. How many ways can you pick out 4 beers to give to your friend Jane and then 2 beers to give to your friend Jack?

a) $\frac{12!}{(12-4)!} \times \frac{12!}{(8-2)!}$

b) $\frac{12!}{4!(12-4)!} \times \frac{12!}{2!(8-2)!}$

c) $\frac{12!}{4!(12-4)!} \times \frac{8!}{2!(8-2)!}$ Correct. [ways to choose 4 of 12 items] $\times$ [ways to choose 2 of the 8 remaining items]

j) You actually have 5 very wide shelves to display your beer. How many ways can you arrange all 99 different bottles across these 5 shelves?

a) $\frac{(99+5)!}{99! \times 5!}$

b) $\frac{(99+5)!}{5!}$

c) $\frac{(99+4)!}{4!}$ Correct. [ways to order 99 different items and partition them into 5 groups (which requires 4 partitions)]

k) It turns out you somehow got 8 identical bottles of "Lugubrious Lager". One bottle of that stuff is really more than you need. How many ways could you distribute those 8 bottles among 4 of your friends?

a) $\frac{8!}{4!}$

b) $\frac{(8+3)!}{3!}$

c) $\frac{(8+3)!}{3!8!}$ Correct. [ways to partition 8 identical items into 4 groups]

l) Let's say you have 50 different american bottles of beer. Specifically, you have 10 bottles from each of 5 different regions of the country (East Coast, South, Midwest, Southwest, and Plains States). You're having a party, and you want to let people sample some things from your collection. How many ways can you pick 2 bottles from each region of the country?

a) $5 \times \frac{10!}{2!(10-2)!}$

b) $(\frac{10!}{2!(10-2)!})^5$ Correct. [ways to choose 2 of 10 items] × [ways to choose 2 of 10 items] × [ways to choose 2 of 10 items] × [ways to chose 2 of 10 items] × [ways to choose 2 of 10 items]

m) Same situation as above, but it turns out only one guy showed up to your party. You'd like to pick just one region and give him two beers to try from it. How many ways can you do this?

a) $5 \times \frac{10!}{2!(10-2)!}$ Correct. [ways to choose 2 of 10 items] + [ways to choose 2 of 10 items] + [ways to choose 2 of 10 items] + [ways to chose 2 of 10 items] + [ways to choose 2 of 10 items]

b) $(\frac{10!}{2!(10-2)!})^5$

n) You have a small cooler for tailgating that will hold at most 5 bottles. Also, you have a superstition that taking an even number of bottles to the game would be bad luck. How many ways can you choose an odd number of your 99 bottles to put in the cooler?

a) $\frac{99!}{5!(99-5)!} \times \frac{99!}{3!(99-3)!} \times 99$

b) $\frac{99!}{5!(99-5)!} + \frac{99!}{3!(99-3)!} + 99$ Correct. [ways to chose 5 of 99 items] + [ways to choose 3 of 99 items] + [ways to choose 1 of 99 items]

4) Answer the following combinatorics problems.

a) Easter Eggs: You're painting Easter Eggs. If you paint 3 big stripes on each egg, and you have 5 colors to choose from (red, blue, yellow, green, purple), then how many visibly different eggs can you make?

[possible colors for top stripe] × [possible colors for middle stripe] × [possible colors for bottom stripe] $= 5 \times 5 \times 5 = 5^3$

b) Dinner: You're eating dinner at a dorm cafeteria. Tonight, there are 4 desert options, 5 entree options, and 6 side item options. **(i)** How many ways can you pick out two deserts? **(ii)** How many ways can you pick out 3 side items? **(iii)** If your meal plan covers 1 entree, 3 side items, and 2 deserts, how many different meals can you order?)

There's two ways to look at this problem, depending on how much you like the cheesecake or the mashed potatoes. If you figured that you could pick up multiples of the same item (so take two plates of cheesecake for your two deserts), then the answers are as follows: **(i)** 4×4, **(ii)** $6 \times 6 \times 6$, **(iii)** $5 \times 6^3 \times 4^2$.

Alternatively, if you assumed that you would only take one of each item, then the answers are as follows: **(i)** $\frac{4!}{2!(4-2)!}$, **(ii)** $\frac{6!}{3!(6-3)!}$, **(iii)** $5 \times \frac{6!}{3!(6-3)!} \times \frac{4!}{2!(4-2)!}$.

c) Applied Problem Sliding 8 Puzzle: A problem that's often taught in AI courses is the Sliding 8 puzzle. This is a 3x3 square containing 8 sliding tiles (with the 9th spot is left open). The tiles are numbered 1 through 8. If a tile is next to the open spot, it can be slid into the open spot, which leaves its old spot open for another tile to slide into, and so on. The goal of the puzzle is to start with the tiles random ly arranged in the square, and then have your AI program find a way to slide the tiles around the square and back into 1-8 order. In order to do this, the program searches through the different possible configurations of the board. How many ways can you arrange 8 different tiles and a blank spot across 9 possible spaces?

If you have 9 things (8 tiles and a blank spot) and 9 spots to put them in, and you want to count the number of ways you can arrange the things in the spots, then there's 9! ways to do that.

d) Coin Flip Problem: **(i)** If you flip a coin 5 times, how many different sequences of Heads and Tails can you get? Think of a given sequence as being a string of H and T symbols (so HTTTT represents the case where only the first flip comes up Heads). **(ii)** How many ways can you get 3 Heads? **iii** How many total ways can you get more Heads than Tails? Go ahead and evaluate the factorials and exponents to find the actual numerical answers for iii and i. **(iv)** How do the answers compare? Why does this make sense?)

(i) [choices for first letter] $\times$ [choices for second letter] $\times$ [choices for third letter] $\times$ [choices for fourth letter] $\times$ [choices for fifth letter.] $= 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$, **(ii)** Number of different strings with 3 H's and 2T's is equal to the number of ways to order 5 things if 3 of them are identical and 2 of them are identical $= \frac{5!}{3!2!}$. **(iii)** [number of strings with 3 H's] + [number of strings with 4 H's] + [number of strings with 5 H's] $= \frac{5!}{3!2!} + \frac{5!}{4!1!} + 1 = 10 + 5 + 1 = 16$. **(iv)** There's 16 ways to get more heads than tails, and 32 total possible ways to flip a coin 5 times. So, exactly half of the possibilities give you more heads than tails, which is what you'd expect from an unbiased coin.

Proofs:

6) Easy Proof: Prove that so long as k and n are both at least 2, there are always more ways to permute k of n items than there are to choose k of n items. (You don't need to use induction here or anything. Just look at the formulas. Then explain clearly and precisely why this is true).

We can look at the two formulas, permute k of n: $\frac{k!}{(k-n)!}$ and choose k of n: $\frac{k!}{k!(k-n)!}$, and you can see that the numerators are identical, but the denominator of the second one will always be larger than the denominator of the first (so long as $k \geq 2$), because it's multiplied by $k!$. So the choose formula will always give you a smaller number. This should make sense intuitively too; since we don't care about order in choose, there's fewer distinct possibilities.

7) Medium Proof: Prove that the number of ways you can choose a group of k items from a set of n items, is the same as the number of ways you can choose a group of (n-k) items from a set of n items. (In other words, if you've got n books, picking k of them to take with you is basically the same as choosing (n-k) of them to leave behind.) Use the formulas to clearly explain why these two things are always equivalent.

$$\begin{pmatrix} n \\ k \end{pmatrix} =^? \begin{pmatrix} n \\ (n-k) \end{pmatrix}$$
$$\frac{n!}{k!(n-k)!} =^? \frac{n!}{(n-k)!(n-(n-k))!}$$
$$\frac{n!}{k!(n-k)!} =^? \frac{n!}{(n-k)!(n-n+k)!}$$
$$\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!}$$

8) (*Extra Credit*) Hard Proof: Problem 5.28 in your book gives an inductive proof of the binomial theorem. To determine the number of subsets of size k that can be picked from a set of n items, we use the choose formula. If we wanted to know the number of subsets that were of size k or of size j, we could use the choose formula for both and then sum the results. So, what would it look like if we were to use the choose formula to find the total number of subsets of *any* size from a set of n items? We've

also discussed a better method of viewing this problem which produces the answer 2^n (what method was this?). This leaves us with two very different looking ways of finding the same value. Use the theorem proved in 5.28 to justify why these two methods produce the same number.

So the 2^n formula comes from phrasing the question of "How many possible subsets can we make from n items?" as: "include item 1 in our subset?" and "include item 2?" and "include item 3"... and "include item n?". Which gives us a series of n decisions which each have 2 options, so we get $2 \times 2 \times 2... = 2^n$

Meanwhile, we could come up with another formula by counting: [ways to choose a subset of 0 items] + [ways to choose a subset of 1 item] + [ways to choose a subset of 2 items] + ...[ways to choose a subset of (n-1) items] + [ways to choose a subset of n items] $= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} ... + \binom{n}{(n-1)} + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}$

So we'd really like it to be true that $2^n = \sum_{i=0}^{n} \binom{n}{i}$. Happily this is one consequence of the Binomial Theorem. If it's always true that $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$, then when $a = b = 1$, we get our desired result of:

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i = \sum_{i=0}^{n} \binom{n}{i}$$