C241 Homework 3: Combinatorics, Basic Propositional Logic

Due Wednesday, 9/23/09

1) Match each numerical value with the description of the quantity that it's counting. It may help to look at the solutions for problem 2) from the previous assignment. For these problems you have 15 textbooks: 5 CS books, 7 math books (including three identical copies of "Calculus I"), 2 physics, and 1 book of arthurian legends.

Values

s) $5! \times 2! \times 1!$

t) $5 + 2 + 1$

u) $5 \times 2 \times 1$

v) $7!$

w) $(7 - 3)!$

x) $\frac{7!}{3!}$

y) $\frac{5!}{(5-3)!}$

z) $\frac{5!}{3!(5-3)!}$

Descriptions

a) Number of ways to arrange 3 of your CS books on a shelf

b) Number of ways to arrange all of your math books which are not copies of "Calculus I" in a line on a shelf.

c) Number of visibly different ways to arrange all of your math books on a shelf

d) Number of ways to order your 5 CS and 2 physics books together on a shelf, with no restrictions.

e) Number of ways to choose a single book of some type besides math, to give to a friend.

f) Number of ways to set everything except your math books on the shelf, with the restriction that the CS books come first, then the physics, and last the book of arthurian legends.

g) Number of ways to pick out 3 of your CS books to loan to a friend.

h) Number of ways to select one book of every type besides math, to loan to a friend.

2) Complex combinatorics problems are solved by breaking them into a series of simple questions and then correctly combining the answers to find the total quantity. Below is a series of complex combinatorics problems and strategies for solving them. Find the answers for all the simple questions, and then combine the answers to get the total quantities for each. The first one's been done for you.

a) In a deck of playing cards there are 13 cards in each of four suits (Clubs, Spades, Hearts, Diamonds), for a total of 52 cards. The 13 cards in each suit are labeled with the values: Ace,2,3,4,5,6,7,8,9,10,Jack,Queen, and King. In poker, a "flush" is a hand that has five cards of the same suit. How many different possible flushes are there?
[number of possible suits] × [number of ways to pick 5 cards from a suit with 13 cards]
This gives us the answer: $4 \times \frac{13!}{5!(13-5)!}$

b) How many ways can 15 (identical) candy bars be distributed among five children so that the youngest only gets one or two of them?.
[(if youngest gets 1) how many ways can 14 identical candy bars be divided among 4 older children] + [(if youngest gets 2) how many ways can 13 identical candy bars be divided among 4 older children?]

c) I'd like to know how may different truth tables I can build with two variables (we'll say two truth tables are the same if the final answers on each line are the same, we're not worried about the work needed to get the final answer here). If I have two variables, then my truth table will have four lines: TT, TF, FT, FF, and the final answer on each of those lines will be either T or F. So I can figure out the total number of possibilites as follows:
[number of possible answers for TT line] × [number of possible answers for TF line] × [number of possible answers for FT line] × [number of possible answers for FF line]

d) In poker, a "full house" is a hand of five cards in which two of the cards all have one value ('two of a kind') and the other three all have a different value ('three of a kind'). For example, one full house would be: Clubs 2, Hearts 2, Diamonds Jack, Spades Jack, Hearts Jack. How many different full house hands

are possible?
[number of possibilities for the 'two of a kind' value] $\times$ [number of ways to choose two different suits for the 'two of a kind' cards] $\times$ [number of possibilities for the 'three of a kind' value] $\times$ [number of ways to choose three different suits for the 'three of a kind' cards]

e) Let's say there are five 300-level undergrad CS classes being offered this semester, and there are 30 undergrad CS majors interested in taking 300-level classes, but each student can enroll in at most three CS classes. How many possible total assignments of students to classes are there?
([number of ways one undergrad can choose 3 of 5 classes to take] + [number of ways one undergrad can choose 2 of 5 classes to take] + [number of ways one undergrad can choose 1 of 5 classes to take])$^{\text{[total number of undergrads]}}$

3) Answer the following combinatorics problems. Explain what strategy you're using for each one, as we did above.

a) "Have it Your Way:" A fast food burger place advertises that a customer can have his or her hamburger *with or without* any or all of the following: catsup, mustard, mayonnaise, lettuce, tomato, onion, pickle, cheese, or mushrooms. How many different kinds of hamburger orders are possible? (Note: the order the toppings are placed on the burger isn't important).

b) In poker, "four of a kind", means a hand of five cards, four of which have the same value (so Clubs Ace, Hearts Ace, Diamonds Ace, Spades Ace, Hearts 3 is "four of a kind"). How many different "four of a kind" hands are possible?

c) At a middle school dance there are 20 girls and 20 boys. How many different ways are there to match the kids into 20 girl/boy couples? Of course, not all of the couples go out on the dance floor for each song... let's call the couples that are actually dancing the "dancing set". How many different dancing sets can there be?

4) Write the truth tables for the following logical statements. You may find the lecture notes posted under resources on the website are useful here.

a) $p \vee p$

b) p

c) $p \wedge \neg(p \vee p)$

d) $p \rightarrow p$

e) $(p \vee \neg p) \rightarrow (q \wedge \neg q)$

f) $(p \vee (\neg q \wedge r))$

g) $(p \vee \neg q) \wedge (p \vee r)$

5) Answer the following questions about the statements above:

a) Which of the statements above is a tautology?

b) Which of the statements above are contradictions?

c) Which pairs of statements above are equivalent to each other?

6) Create four new logical statements yourself (you can't use any of the ones from problem 4 or any of the equivalence laws). The first should be a contradiction. The second one should be a tautology. The third and fourth should be equivalent to each other and should be neither contradictions nor tautologies. Write the truth tables for your statements to show that they have the properties you claim.

7) Attached to this homework is a list of basic, useful logical equivalences called the "Algebraic Laws of Logic". From this list, and from the truth tables above, I know that $(p \vee p) \equiv p$.

(a) Explain briefly in your own words why this equivalence also implies that $r \wedge (p \vee p) \equiv r \wedge p$ (why would you expect these two statements to also have the same truth tables?) Replacing a piece of a logical statement with something that's equivalent to it is called "substitution".

(b) Explain briefly in your own words why this equivalence also implies that $(q \wedge r) \vee (q \wedge r) \equiv (q \wedge r)$ (why would you expect these two statements to have the same truth tables?). The logical laws are properly written as $(P \vee P) \equiv P$, where P is a variable that can represent any logical statement.

8) Below are two examples of using our list of equivalence laws to simplify a complex logical statement. The first example is complete. For the second example, fill in the names of the rules that are used at each step, following the same style as the first example. Square brackets are just used for clarity, they mean the same thing as regular parentheses.

a) $(p \to q) \wedge [\neg q \wedge (r \vee \neg q)]$

$\equiv (p \to q) \wedge \neg q$	Second Absorption Law
$\equiv (\neg p \vee q) \wedge \neg q$	Definition of $\to$
$\equiv \neg q \wedge (\neg p \vee q)$	Second Commutative Law
$\equiv (\neg q \wedge \neg p) \vee (\neg q \wedge q)$	Second Distributive Law
$\equiv (\neg q \wedge \neg p) \vee F$	Second Complement Law
$\equiv (\neg q \wedge \neg p)$	First Identity Law
$\equiv \neg(q \vee p)$	First DeMorgan's Law

b) $[(p \vee q) \wedge (p \vee \neg q)] \vee q$

$\equiv [p \vee (q \wedge \neg q)] \vee q$	________________
$\equiv (p \vee F) \vee q$	________________
$\equiv p \vee q$	________________

Proofs:

9) Easy Proof: Use truth tables to prove that $\neg(p \vee q)$ is *not* equivalent to $\neg p \vee \neg q$, and that $\neg(p \wedge q)$ is *not* equivalent to $\neg p \wedge \neg q$. Make sure that you both write the truth tables and briefly explain in english how the truth tables demonstrate that the two statements are not equivalent.

10) Medium Proof: Explain why, if I have a list of 17 different logical statements that use the two variables p and q, there's at least two of them that are equivalent to each other (in other words, at least two of them have to have the same truth table). (Your answer to problem 2c and the fact that $17 = 2^4 + 1$ may be useful here). Try to make your explanation as clear and easy to understand as possible.

11)(*Extra Credit*) Hard Proof: (a) Using *only* the equivalence laws of logic on the attached page, prove that DeMorgan's conjunctive rule is valid for three variables as well as two. In other words, prove that $\neg(p \wedge q \wedge r) \equiv (\neg p \vee \neg q \vee \neg r)$ (b) Use induction to prove that DeMorgan's rule is valid for n variables, for any natural number n. In other words, prove that $\neg(p_1 \wedge p_2 \wedge ... p_{n-1} \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee ... \neg p_{n-1} \vee \neg p_n)$. (Obviously, the analogous proofs for DeMorgan's disjunctive rule would be very similar.)

A Chart of the Algebraic Laws of Logic

(An extension of the chart on page 75 in your textbook)

Note: The symbols P,Q,R,K below can represent complex logical statements.

Double Negation	$\neg(\neg P) \equiv P$
Identity Laws	$(P \vee F) \equiv P$ $(P \wedge T) \equiv P$ $(P \vee T) \equiv T$ $(P \wedge F) \equiv F$
Complement Laws	$(P \vee \neg P) \equiv T$ $(P \wedge \neg P) \equiv F$
Idempotence Laws	$(P \vee P) \equiv P$ $(P \wedge P) \equiv P$
Commutative Laws	$(P \vee Q) \equiv (Q \vee P)$ $(P \wedge Q) \equiv (Q \wedge P)$
Associative Laws	$(P \vee (Q \vee R)) \equiv ((P \vee Q) \vee R) \equiv (P \vee Q \vee R)$ $(P \wedge (Q \wedge R)) \equiv ((P \wedge Q) \wedge R) \equiv (P \wedge Q \wedge R)$
DeMorgan's Laws	$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$ $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$
Distributive Laws	$P \vee (K \wedge Q) \equiv (P \vee K) \wedge (P \vee Q)$ $P \wedge (K \vee Q) \equiv (P \wedge K) \vee (P \wedge Q)$
Subsumption Laws	$P \vee (P \wedge Q) \equiv P$ $P \wedge (P \vee Q) \equiv P$
Definition of Implication	$(P \rightarrow Q) \equiv (\neg P \vee Q)$