

## C241 Homework 4: Simplifications, Inference Proofs and Predicates

Due Wednesday, 9/30/09

1) Simplify the following logical statements, using the algebraic laws of logic (label each step with the law you use). The second one is a tautology, so it's possible to simplify it all the way to the value *True*. The equivalence  $(p \rightarrow q) \equiv (\neg p \vee q)$  will be useful for it.

a)  $(\neg p \wedge (p \vee q)) \vee (q \wedge \neg q)$

b)  $((p \vee q) \rightarrow p) \rightarrow (q \rightarrow p)$

2) The following are three different ways of stating the same thing:

$$\begin{array}{l} P \rightarrow Q \\ * \quad P \\ \hline \therefore Q \end{array}$$

\*  $(P \wedge (P \rightarrow Q)) \Rightarrow Q$

\*  $(P \wedge (P \rightarrow Q))$  logically implies  $Q$

a) If  $R$  and  $S$  are two complex logical statements, what's the difference between stating that  $R \equiv S$  and  $R \Rightarrow S$ ?

b) If  $R \equiv S$  then on every truth table line where  $R$  evaluates to True,  $S$  also evaluates to True. And on every truth table line where  $R$  evaluates to False,  $S$  also evaluates to False. In other words, their truth tables match. What if instead, we only knew that  $R \Rightarrow S$ ? What would that imply about the relationship between  $R$ 's and  $S$ 's truth tables?

c) What's wrong with the statement below?

$$(P \vee Q) \wedge \neg P \equiv Q \quad \text{Rule of Disjunctive Simplification}$$

3) Label each step of the logical proof below with the justification for that step. Each of your labels should be one of the following: "premise", one of the logical equivalence laws from the chart at the end of this assignment sheet (Look carefully! There is a new law for  $(P \rightarrow Q)$  ), or a logical inference rule from the other chart at the end of this assignment sheet. Make sure you include the proof lines you're referencing in your label; the label for line (4) has been filled in as an example.

$$\begin{array}{l}
 (\neg p \vee q) \rightarrow r \\
 r \rightarrow (s \vee t) \\
 \neg s \wedge \neg u \\
 \neg u \rightarrow \neg t \\
 \hline
 \therefore p
 \end{array}$$

- 1)  $\neg s \wedge \neg u$
- 2)  $\neg u$
- 3)  $\neg u \rightarrow \neg t$
- 4)  $\neg t$                       lines (2)(3) and Modus Ponens
- 5)  $\neg s$
- 6)  $\neg s \wedge \neg t$
- 7)  $r \rightarrow (s \vee t)$
- 8)  $\neg(s \vee t) \rightarrow \neg r$
- 9)  $(\neg s \wedge \neg t) \rightarrow \neg r$
- 10)  $\neg r$
- 11)  $(\neg p \vee q) \rightarrow r$
- 12)  $\neg r \rightarrow \neg(\neg p \vee q)$
- 13)  $\neg r \rightarrow (p \wedge \neg q)$
- 14)  $p \wedge \neg q$
- 15)  $\therefore p$

4) Do the following logical proofs on your own, using the same style as the proof in problem 2.

a)

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \neg r \\ \hline \therefore \neg(p \vee r) \end{array}$$

b) Do this as a proof by Contradiction. Add  $\neg(s \vee t)$  to the list of premises (i.e., assume the conclusion is false) and then continue the proof until you can get 'False' as a conclusion (which shows that if  $(s \vee t)$  *wasn't* true, then something False would be true, which is a contradiction).

$$\begin{array}{l} p \\ p \rightarrow q \\ s \vee r \\ r \rightarrow \neg q \\ \hline \therefore s \vee t \end{array}$$

5) For this problem, use the following propositions: d="it's day", n="it's night", t="you're in the twilight zone", s="you hate sunlight", v="you are a vampire". Also remember that  $p \rightarrow q$  is read in english as: "if p, then q", and  $p \leftrightarrow q$  is read as: "p if and only if q".

a) Translate these logical statements into english.

- \*  $(d \wedge \neg n) \vee (\neg n \wedge d)$
- \*  $v \rightarrow s$
- \*  $\neg s \rightarrow \neg v$

b) Translate these english sentences into logical statements.

- \* If it's both day and night, then you're in the twilight zone
- \* If you hate sunlight, then you're a vampire
- \* You hate sunlight if and only if you're a vampire

5) For this problem, use the following predicates. The domain of  $x$  is cookies, the domain of  $y$  is ovens, and the domain of  $z$  is people.  $O(x)$  = "x is an oatmeal cookie",  $R(x)$  = "x has raisins",  $C(x)$  = "x is a chocolate chip cookie",  $N(x)$  = "x has nuts",  $B(x, y)$  = "x was baked in y",  $E(z, x)$  = "z ate x". Also remember that  $\exists x : P(x)$  means: "There is at least one  $x$  such that  $P(x)$  is true" (for example  $\exists x : C(x)$  = "There is a chocolate chip cookie"), and  $\forall x : P(x)$  means: "P(x) is true for every x" (so  $\forall x : C(x)$  = "All the cookies are chocolate chip cookies").

a) Translate these logical statements into english.

- \*  $\exists x : C(x) \wedge \neg N(x)$
- \*  $\forall x : O(x) \rightarrow R(x)$
- \*  $\forall x \exists y : B(x, y)$
- \*  $\neg(\exists z \forall x : E(z, x))$
- \*  $\forall x \forall z : E(z, x) \rightarrow \exists y : B(x, y)$

b) Translate these english sentences into logical statements.

- \* There's a cookie which has raisins.
- \* It's not true that all cookies don't have raisins.
- \* There's someone who only ate chocolate chip cookies (In other words, there is a person who, for every cookie, if she ate the cookie then the cookie was chocolate chip).

7) Predicates and quantifiers are incredibly useful tools for precisely defining concepts. We'll be using them throughout the rest of the class, even when we're not doing formal logic. But they're only useful if you really understand what they mean. Below are several questions about pie. Answer each question below with "yes", "no", or "maybe (not enough information)". Think *very* carefully about your answer, you should use 'maybe' frequently.

The domain of  $x$  is students.

$P(x)$  = "x has a pie",  $A(x)$  = "x has an apple pie"

- a) If it's true that:  $\exists x : P(x)$ , then:
- i) Do all students have pies?
  - ii) Does some student have a pie?
- b) If it's true that:  $\forall x : A(x)$ , then:
- i) Does some student have an apple pie?
  - ii) Do all students have apple pies?
  - iii) If a pie belongs to a student, is it an apple pie?
- c) If it's true that:  $\forall x : \neg A(x)$ , then:
- i) Does some student have an apple pie?
  - ii) Does some student have a pie?
- d) If it's true that:  $\neg \forall x : A(x)$ , then:
- i) Does every student have an apple pie?
  - ii) Does any student have an apple pie?
- e) If it's true that:  $\neg \exists x : P(x)$ , then:
- i) Does every student have a pie?
  - ii) Does any student have a pie?
- f) If it's true that:  $\forall x : P(x) \wedge A(x)$
- i) Is there any student who doesn't have a pie?
  - ii) If a pie belongs to a student, is it an apple pie?
  - iii) Does every student have two pies?
- g) If it's true that:  $\forall x : P(x) \rightarrow A(x)$
- i) Do all students have pies?
  - ii) Does some student have a pie?
  - iii) If a student has only one pie, is it an apple pie?
- h) If it's true that:  $\exists x : P(x) \vee A(x)$
- i) Does some student have a pie?
  - ii) Does some student have an apple pie?
  - iii) Do all student have pies?

8) Fill in the blanks, by writing the name of the rule used next to each line of this formal proof.

$$\begin{array}{l}
 \exists x : \neg r(x) \\
 \forall x : (p(x) \vee r(x)) \\
 \forall x : (p(x) \rightarrow q(x)) \\
 \hline
 \therefore \exists x : q(x)
 \end{array}$$

- |    |                                       |   |
|----|---------------------------------------|---|
| 1) | $\exists x : \neg r(x)$               | premise   |
| 2) | $\neg r(c)$                           | Existential Specification (1)   |
| 3) | $\forall x : (p(x) \vee r(x))$        | _____   |
| 4) | $p(c) \vee r(c)$                      | Universal Specification (2)(3) (since $(p(x) \vee r(x))$<br>is true for all $x$ , we know it's true for the specific case $c$ ) |
| 5) | $\forall x : (p(x) \rightarrow q(x))$ | _____   |
| 6) | $p(c) \rightarrow q(c)$               | _____   |
| 7) | $p(c)$                                | _____   |
| 8) | $q(c)$                                | _____   |
| 9) | $\exists x : q(x)$                    | _____   |

Proofs:

9) **Easy Proofs:** There are two useful plain-english proof techniques we can talk about more clearly now that we have quantifiers. The first one is proof by construction. If I ask you to prove that  $\exists x : P(x)$ , all you have to do is provide me with an example of an  $x$  that has property  $P$ . The second useful technique is proof by counter-example. If I ask you to prove that  $\neg\forall x : Q(x)$ , all you have to do is give me an example of an  $x$  that *doesn't* have property  $Q$ . Try this:

- a) Prove by construction that  $\exists x : (x \text{ is even}) \wedge (x > 10)$ , where the domain of  $x$  is integers.
- b) Give a counter-example to show that  $\neg\forall x : (x < 13)$ , where the domain of  $x$  is integers.

10) **Hard Proof (Bonus):** It turns out we don't really need all the logical operators we defined. We can actually write some logical operators in terms of other ones. For instance, I can write  $(p \vee q)$  using just  $\wedge$  and  $\neg$ . The statement  $\neg(\neg p \wedge \neg q)$  will produce the same truth table as  $(p \vee q)$ . In fact, I can create statements which will produce the truth tables for all the logical operators  $\wedge, \vee, \neg, \rightarrow$  using only the "NAND" operator defined below. Can you?

$p$	$q$	$p \text{ NAND } q$
F	F	T
F	T	F
T	F	F
T	T	F

## A Chart of the Algebraic Laws of Logic

*Note: The symbols  $P, Q, R, K$  below can represent complex logical statements.*

<b>Double Negation</b>	$\neg(\neg P) \equiv P$
<b>Identity Laws</b>	$(P \vee F) \equiv P$ $(P \wedge T) \equiv P$ $(P \vee T) \equiv T$ $(P \wedge F) \equiv F$
<b>Complement Laws</b>	$(P \vee \neg P) \equiv T$ $(P \wedge \neg P) \equiv F$
<b>Idempotence Laws</b>	$(P \vee P) \equiv P$ $(P \wedge P) \equiv P$
<b>Commutative Laws</b>	$(P \vee Q) \equiv (Q \vee P)$ $(P \wedge Q) \equiv (Q \wedge P)$
<b>Associative Laws</b>	$(P \vee (Q \vee R)) \equiv ((P \vee Q) \vee R) \equiv (P \vee Q \vee R)$ $(P \wedge (Q \wedge R)) \equiv ((P \wedge Q) \wedge R) \equiv (P \wedge Q \wedge R)$
<b>DeMorgan's Laws</b>	$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$ $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$
<b>Distributive Laws</b>	$P \vee (K \wedge Q) \equiv (P \vee K) \wedge (P \vee Q)$ $P \wedge (K \vee Q) \equiv (P \wedge K) \vee (P \wedge Q)$
<b>Subsumption Laws</b>	$P \vee (P \wedge Q) \equiv P$ $P \wedge (P \vee Q) \equiv P$
<b>Definition of Implication</b>	$(P \rightarrow Q) \equiv (\neg P \vee Q)$
<b>Contrapositive</b>	$(P \rightarrow Q) \equiv (\neg Q \rightarrow \neg P)$
<b>Quantifier Negation</b>	$\neg \exists x : P(x) \equiv \forall x : \neg P(x)$ $\neg \forall x : P(x) \equiv \exists x : \neg P(x)$

## A Chart of the Inference Rules of Logic

*Note: The symbols  $P, Q, R, S$  below can represent complex logical statements.  $F = \text{"False"}$*

<b>Modus Ponens</b> $\frac{P \quad P \rightarrow Q}{\therefore Q}$	<b>Law of Syllogism</b> $\frac{P \rightarrow Q \quad Q \rightarrow R}{\therefore P \rightarrow R}$
<b>Modus Tollens</b> $\frac{P \rightarrow Q \quad \neg Q}{\therefore \neg P}$	<b>Rule of Conjunction</b> $\frac{P \quad Q}{\therefore P \wedge Q}$
<b>Rule of Disjunctive Syllogism</b> $\frac{P \vee Q \quad \neg P}{\therefore Q}$	<b>Rule of Contradiction</b> $\frac{\neg P \rightarrow F}{\therefore P}$
<b>Rule of Conjunctive Simplification</b> $\frac{P \wedge Q}{\therefore P}$	<b>Rule of Disjunctive Amplification</b> $\frac{P}{\therefore P \vee Q}$
<b>Rule of Conditional Proof</b> $\frac{P \wedge Q \quad P \rightarrow (Q \rightarrow R)}{\therefore R}$	<b>Rule for Proof by Resolution</b> $\frac{P \vee Q \quad \neg P \vee R}{\therefore (Q \vee R)}$
<b>Rule of the Constructive Dilemma</b> $\frac{P \rightarrow Q \quad R \rightarrow S \quad P \vee R}{\therefore Q \vee S}$	<b>Rule of the Destructive Dilemma</b> $\frac{P \rightarrow Q \quad R \rightarrow S \quad \neg Q \vee \neg S}{\therefore \neg P \vee \neg R}$
<b>Rule of Universal Specification</b> (this is valid for any c) $\frac{\forall x : P(x)}{\therefore P(c)}$	<b>Rule of Universal Generalization</b> $\frac{P(a) \text{ [a is an arbitrary element]}}{\therefore \forall x : P(x)}$
<b>Rule of Existential Specification</b> (c is the specific example) $\frac{\exists x : P(x)}{\therefore P(c)}$	<b>Rule of Existential Generalization</b> $\frac{P(c) \text{ [c is a specific example]}}{\therefore \exists x : P(x)}$