C241 Homework 4: Simplifications, Inference Proofs and Predicates

Due Wednesday, 9/30/09

1) Simplify the following logical statements, using the algebraic laws of logic (label each step with the law you use). The second one is a tautology, so it's possible to simplify it all the way to the value $True$. The equivalence $(p \rightarrow q) \equiv (\neg p \vee q)$ will be useful for it.

a) $(\neg p \wedge (p \vee q)) \vee (q \wedge \neg q)$

$((\neg p \wedge p) \vee (\neg p \wedge q)) \vee (q \wedge \neg q)$ Distributive

$(F \vee (\neg p \wedge q)) \vee (q \wedge \neg q)$ Complement

$(\neg p \wedge q) \vee (q \wedge \neg q)$ Identity

$(\neg p \wedge q) \vee F$ Complement

$(\neg p \wedge q)$ Identity

b) $((p \vee q) \rightarrow p) \rightarrow (q \rightarrow p)$

$(\neg(p \vee q) \vee p) \rightarrow (q \rightarrow p)$ Def. of →

$((\neg p \wedge \neg q) \vee p) \rightarrow (q \rightarrow p)$ DeMorgan's

$((\neg p \vee p) \wedge (\neg q \vee p)) \rightarrow (q \rightarrow p)$ Distributive

$((T \wedge (\neg q \vee p)) \rightarrow (q \rightarrow p)$ Complement

$(\neg q \vee p) \rightarrow (q \rightarrow p)$ Identity

$(\neg q \vee p) \rightarrow (\neg q \vee p)$ Def. of →

$\neg(\neg q \vee p) \vee (\neg q \vee p)$ Def. of →

T Complement

2) The following are three different ways of stating the same thing:

$$* \quad \begin{array}{l} P \rightarrow Q \\ \underline{P \qquad\quad} \\ \therefore Q \end{array}$$

$*$ $(P \wedge (P \rightarrow Q)) \Rightarrow Q$

* $(P \wedge (P \rightarrow Q))$ *logically implies* Q

a) If R and S are two complex logical statements, what's the difference between stating that $R \equiv S$ and $R \Rightarrow S$?

If $R \equiv S$, then they are equivalent: if one is true, the other's true and if one is false, the other's false. Meanwhile, what $R \Rightarrow S$ means is: whenever the statement R is true, then the statement S will also be true (but if R is false, S might still be true, since they're not necessarily equivalent). Note that $R \Rightarrow S$ is not the same as $R \rightarrow S$. $R \rightarrow S$ is just the logical statement which we used on the last assignment; just like $\wedge$ and $\vee$, it can evaluate to either true or false depending on the values of R and S. Meanwhile, when we write $R \Rightarrow S$ we're stating that a particular *relationship* holds between the statements R and S: specifically that whenever R evaluates to true, S does also (similar to the relationship $R \equiv S$, which means that the two statements always evaluate to the same value). We can't *evaluate* the claim $R \Rightarrow S$, it's a relationship that either holds or doesn't hold between R and S. There is of course a reason why the symbols for $\rightarrow$ and $\Rightarrow$ look similar: if we know that the relationship $R \Rightarrow S$ holds between the two statements R and S, then we also know that the logical statement $R \rightarrow S$ will always evaluate to true.
An interesting point: we have $R \Rightarrow T$ for *any* statement R (since T is guaranteed to always be true whenever R is), and also that $F \Rightarrow S$ for any statement S (since all S needs to do is be true whenever F is, which is a very easy thing to do, since F is never true).

b) If $R \equiv S$ then on every truth table line where R evaluates to True, S also evaluates to True. And on every truth table line where R evaluates to False, S also evaluates to False. In other words, their truth tables match. What if instead, we only knew that $R \Rightarrow S$? What would that imply about the relationship between R's and S's truth tables?
On every line that R evaluates to true, S also evaluates to true.

c)What's wrong with the statement below?
$(P \vee Q) \wedge \neg P \equiv Q$ Rule of Disjunctive Simplification
Well, two things. There isn't a rule of Disjunctive Simplification, the above is an attempt to apply the rule of Disjunctive Syllogism. But even so, it's still wrong. Disjunctive Syllogism is an inference rule, it tells us that $(P \vee Q) \wedge \neg P \Rightarrow Q$.

It doesn't tell us that the two statements are equivalent. In fact, after a couple steps of equivalence laws, we find that $(P \vee Q) \wedge \neg P \equiv \neg P \wedge Q$.

3) Label each step of the logical proof below with the justification for that step. Each of your labels should be one of the following: "premise", one of the logical equivalence laws from the chart at the end of this assignment sheet (Look carefully! There is a new law for $(P \rightarrow Q)$), or a logical inference rule from the other chart at the end of this assignment sheet. Make sure you include the proof lines you're referencing in your label; the label for line (4) has been filled in as an example.

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \neg u \rightarrow \neg t \\ \hline \therefore p \end{array}$$

1)	$\neg s \wedge \neg u$	premise
2)	$\neg u$	line (1) and Conjunctive Simplification
3)	$\neg u \rightarrow \neg t$	premise
4)	$\neg t$	lines (2)(3) and Modus Ponens
5)	$\neg s$	line (1) and Conjunctive Simplification
6)	$\neg s \wedge \neg t$	lines (4)(5) and Rule of Conjunction
7)	$r \rightarrow (s \vee t)$	premise
8)	$\neg(s \vee t) \rightarrow \neg r$	line (7) and $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$ (Contrapositive)
9)	$(\neg s \wedge \neg t) \rightarrow \neg r$	line (8) and DeMorgans
10)	$\neg r$	lines(6)(9) and Modus Ponens
11)	$(\neg p \vee q) \rightarrow r$	premise
12)	$\neg r \rightarrow \neg(\neg p \vee q)$	line (11) and $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$ (Contrapositive)
13)	$\neg r \rightarrow (p \wedge \neg q)$	line (12) and DeMoragans, and Double Negation
14)	$p \wedge \neg q$	lines (10)(13) and Modus Ponens
15)	$\therefore p$	line (14) and Conjunctive Simplification

4) Do the following logical proofs on you own, using the same style as the proof in problem 2.

a)

$p \to q$
$\neg q$
$\neg r$
$\therefore \neg(p \lor r)$

1.$p \to q$	premise
2.$\neg q$	premise
3.$\neg p$	Modus Tollens (1)(2)
4.$\neg p \land \neg q$	Rule of Conjunction (2)(3)
5.$\neg(p \lor q)$	DeMorgan’s (4)

b) Do this as a proof by Contradiction. Add $\neg(s \lor t)$ to the list of premises (i.e., assume the conclusion is false) and then continue the proof until you can get ’False’ as a conclusion (which shows that if $(s \lor t)$ *wasn’t* true, then something False would be true, which is a contradiction).

p
$p \to q$
$s \lor r$
$r \to \neg q$
$\therefore s \lor t$

1,$\neg(s \lor t)$	premise
2.$\neg s \land \neg t$	DeMorgans (1)
3.$\neg s$	Rule of Conjunctive Simplification (2)
4.$s \lor r$	premise
5.r	Rule of Disjunctive Syllogism (3,4)
6. $r \to \neg q$	premise
7.$\neg q$	Modus Ponens (5,6)
8.$p \to q$	premise
9.$\neg p$	Modus Tollens (7,8)
10.p	premise
11.$\neg p \land p$	Rule of Conjunction (9,10)
12.$False$	Contradiction (11)

5)For this problem, use the following propositions: d="it's day", n="it's night", t="you're in the twilight zone", s="you hate sunlight", v="you are a vampire". Also remember that $p- > q$ is read in english as: "if p, then q", and $p \leftrightarrow q$ is read as: "p if and only if q".

a) Translate these logical statements into english.

- $(d \wedge \neg n) \vee (\neg n \wedge d)$
  "It is day and it's not night, or it's not night and it is day"

- $v \rightarrow s$
  "If you're a vampire, then you hate sunlight"

- $\neg s \rightarrow \neg v$
  "If you don't hate sunlight, you're not a vampire"

b) Translate these english sentences into logical statements.

- If it's both day and night, then you're in the twilight zone
  $(d \wedge n) \rightarrow t$

- If you hate sunlight, then you're a vampire
  $s \rightarrow v$

- You hate sunlight if and only if you're a vampire
  $s \leftrightarrow v$

5)For this problem, use the following predicates. The domain of x is cookies, the domain of y is ovens, and the domain of z is people. $O(x) =$ "x is an oatmeal cookie", $R(x) =$ "x has raisins", $C(x) =$ "x is a chocolate chip cookie", $N(x) =$ "x has nuts", $B(x, y) =$ "x was baked in y", $E(z, x) =$"z ate x". Also remember that $\exists x : P(x)$ means: "There is at least one x such that $P(x)$ is true" (for example $\exists x : C(x) =$"There is a chocolate chip cookie"), and $\forall x : P(x)$ means: "P(x) is true for every x" (so $\forall x : C(x)$="All the cookies are chocolate chip cookies").

a) Translate these logical statements into english.

- $\exists x : C(x) \land \neg N(x)$
  "There is a cookie which chocolate and doesn't have nuts." (Or, in more normal english: "One of the cookies is chocolate chip and doesn't have nuts").

- $\forall x : O(x) \rightarrow R(x)$
  "For all cookies, if the cookie is oatmeal then it has raisins". (Or, "All oatmeal cookies have raisins")

- $\forall x \exists y : B(x, y)$
  "For every cookie, there's an oven such that the cookie was baked in that oven". (Or, "Every cookie was baked in an oven")

- $\neg(\exists z \forall x : E(z, x))$
  "It's not true that there's one person such that, for every cookie, that person ate the cookie". (Or, "No one person ate all the cookies").

- $\forall x \forall z : E(z, x) \rightarrow \exists y : B(x, y)$
  "For every cookie and every person, if a person ate a cookie then there is an oven that the cookie was baked in" (Or "Every cookie that was eaten was baked in an oven").

b) Translate these english sentences into logical statements.

- There's a cookie which has raisins.
  $\exists x : R(x)$

- It's not true that all cookies don't have raisins.
  $\neg(\forall x : \neg R(x))$ (which is equivalent to $\exists x : R(x)$.

- There's someone who only ate chocolate chip cookies (In other words, there is a person who, for every cookie, if she ate the cookie then the cookie was chocolate chip).
  $\exists z \forall x : E(z, x) \rightarrow C(x)$
  (Note, this is not the same thing as: $\forall x \exists z : E(z, x) \rightarrow C(x)$. If we translate this logical statement into english, we get "For each cookie, there is some person, such that if the person ate that cookie then the cookie was chocolate chip"... which is a much weirder situation, suggesting that every cookie can magically transform into chocolate chip if eaten by the right person.)

7) Predicates and quantifiers are incredibly useful tools for precisely defining concepts. We'll be using them throughout the rest of the class, even when we're not doing formal logic. But they're only useful if you really understand what they mean. Below are several questions about pie. Answer each question below with "yes, "no", or "maybe (not enough information)". Think *very* carefully about your answer, you should use 'maybe' frequently.

The domain of x is students.

$P(x) =$"x has a pie", $A(x) =$"x has an apple pie"

a) If it's true that: $\exists x : P(x)$, **then:**

(M) i) Do all students have pies?

(If all students did have pies, then we can find an example of a student with a pie. On the other hand, that would also be true if only one student had a pie).

(Y) ii) Does some student have a pie?

(This statement tells us that we can find an example of a student with a pie).

b) If it's true that: $\forall x : A(x)$, **then:**

(Y) i) Does some student have an apple pie?

(This statement tells us that all students have apple pies, so certainly we can find an example of a student with an apple pie).

(Y) ii) Do all students have apple pies?

(Yes, that's what this statement means)

(M) iii) If a pie belongs to a student, is it an apple pie?

(All students have apple pies, but a student could have a strawberry pie in addition to his apple pie.)

c) If it's true that: $\forall x : \neg A(x)$, **then:**

(N) i) Does some student have an apple pie?

(This statement tells that all students don't have apple pies, or equivalently, that we cannot find an example of a student with an apple pie: $\neg \exists x : A(x)$).

(M) ii) Does some student have a pie?

(Just because no one has an apple pie, doesn't mean someone couldn't have a pecan pie).

d) If it's true that: $\neg\forall x : A(x)$, then:

(N) i) Does every student have an apple pie?

(This statement tells us that not everyone has an apple pie (it's false that everyone has an apple pie). That means at least someone doesn't have an apple pie (if we move the negation inside): $\exists x : \neg A(x)$).

(M) ii) Does any student have an apple pie?

(It's possible. All this statement says is that at least one student doesn't have an apple pie. It could be that no one has an apple pie, or it could be that everyone but bob has an apple pie. We don't know).

e) If it's true that: $\neg\exists x : P(x)$, then:

(N) i) Does every student have a pie?

(This statement says that we can't find an example of a student with a pie. Thus *no one* has a pie).

(N) ii) Does any student have a pie?

(Nope. No one has a pie).

f) If it's true that: $\forall x : P(x) \wedge A(x)$ then:

(N) i) Is there any student who doesn't have a pie?

(This statement says that all students have pies and all students have apple pies. Thus we know that all students have pies).

(M) ii) If a pie belongs to a student, is it an apple pie?

(We know that all students have pies, and that all students have apple pies. But a student *could* have both an apple pie and a banana creme pie. Just because everyone has an apple pie doesn't mean that all pies are apple).

(M) iii) Does every student have two pies?

(Maybe. But all we know from this statement is that everyone has at least one apple pie.)

g) If it's true that: $\forall x : P(x) \rightarrow A(x)$ then:

(M)i) Do all students have pies?

(This statement says that, for all students, *if* a student has a pie, then that student has an apple pie. However, it doesn't claim that any student actually *does* have a pie, it just says what would be true *if* a student had a pie.)

(M) ii) Does some student have a pie?
(It's actually possible that no one has a pie)

(Y) iii) If a student has only one pie, is it an apple pie?
(Yes, if a student has a pie, then that student has an apple pie. And if the student only has *one* pie, then that pie must *be* the apple pie.)

h) If it's true that: $\exists x : P(x) \vee A(x)$ **then:**

(Y) i) Does some student have a pie?
(This statement says that we can find an example of a student who either has a pie or has an apple pie. So at least someone out there has a pie).

(M) ii) Does some student have an apple pie?
(If there was only one student, with a single blueberry pie, this statement would still be true... since that student would be an example of an x that satisfied $P(x)$, and if $P(x)$ is true, then $P(x) \vee A(x)$ is true).

(M) iii) Do all student have pies?
(If everyone had a pie, then we could easily find an example of a student with a pie).

8) Fill in the blanks, by writing the name of the rule used next to each line of this formal proof.

$\exists x : \neg r(x)$
$\forall x : (p(x) \vee r(x))$
$\forall x : (p(x) \rightarrow q(x))$

$\therefore \exists x : q(x)$

1)	$\exists x : \neg r(x)$	premise
2)	$\neg r(c)$	Existential Specification (1)
3)	$\forall x : (p(x) \vee r(x))$	premise
4)	$p(c) \vee r(c)$	Universal Specification (2)(3) (since $(p(x) \vee r(x)$ is true for all x, we know it's true for the specific case c)
5)	$\forall x : (p(x) \rightarrow q(x))$	premise
6)	$p(c) \rightarrow q(c)$	Universal Specification (5) (same value c that we used in (2))
7)	$p(c)$	disjunctive syllogism (2)(4)
8)	$q(c)$	modus ponens (6)(7)
9)	$\exists x : q(x)$	Existential Generalization

Proofs:

9) Easy Proofs: There are two useful plain-english proof techniques we can talk about more clearly now that we have quantifiers. The first one is proof by construction. If I ask you to prove that $\exists x : P(x)$, all you have to do is provide me with an example of an x that has property P. The second useful technique is proof by counter-example. If I ask you to prove that $\neg\forall x : Q(x)$, all you have to do is give me an example of an x that *doesn't* have property Q. Try this:

a) Prove by construction that $\exists x : (x \textit{ is even}) \wedge (x > 10)$, where the domain of x is integers.

12 is an example of an integer which is even and greater than 10.

b) Give a counter-example to show that $\neg\forall x : (x < 13)$, where the domain of x is integers.

14 is an example of an integer which is greater than 13.

10) Hard Proof (Bonus): It turns out we don't really need all the logical operators we defined. We can actually write some logical operators in terms of other ones. For instance, I can write $(p \vee q)$ using just $\wedge$ and $\neg$. The statement $\neg(\neg p \wedge \neg q)$ will produce the same truth table as $(p \vee q)$. In fact, I can create statements which will produce the truth tables for all the logical operators $\wedge, \vee, \neg, \rightarrow$ using only the "NOR" operator defined below. Can you?

Note: the original problem called this operator the NAND operator, which is incorrect. This one's NOR. NAND is equivalent to $\neg(p \wedge q)$. But, NOR also works fine. If you knew what NAND actually is and used it correctly in your proof, then you got full credit.

p q	p NOR q
F F	**T**
F T	**F**
T F	**F**
T T	**F**

Try out the truth tables yourself if you'd like to verify these:

$(P\ NOR\ P) \equiv \neg P$

$((P\ NOR\ Q)\ NOR\ (P\ NOR\ Q)) \equiv (P \vee Q)$

$((P\ NOR\ P)\ NOR\ (Q\ NOR\ Q)) \equiv (P \wedge Q)$

$(((P\ NOR\ P)\ NOR\ Q)\ NOR\ ((P\ NOR\ P)\ NOR\ Q)) \equiv (P \rightarrow Q)$ (since $(P \rightarrow Q) \equiv (\neg P \vee Q)$).

A Chart of the Algebraic Laws of Logic

Note: The symbols P,Q,R,K below can represent complex logical statements.

Double Negation	$\neg(\neg P)$	$\equiv$	P		.
Identity Laws	$(P \vee F)$	$\equiv$	P		
	$(P \wedge T)$	$\equiv$	P		
	$(P \vee T)$	$\equiv$	T		
	$(P \wedge F)$	$\equiv$	F		
Complement Laws	$(P \vee \neg P)$	$\equiv$	T		
	$(P \wedge \neg P)$	$\equiv$	F		
Idempotence Laws	$(P \vee P)$	$\equiv$	P		
	$(P \wedge P)$	$\equiv$	P		
Commutative Laws	$(P \vee Q)$	$\equiv$	$(Q \vee P)$		
	$(P \wedge Q)$	$\equiv$	$(Q \wedge P)$		
Associative Laws	$(P \vee (Q \vee R))$	$\equiv$	$((P \vee Q) \vee R)$	$\equiv$	$(P \vee Q \vee R)$
	$(P \wedge (Q \wedge R))$	$\equiv$	$((P \wedge Q) \wedge R)$	$\equiv$	$(P \wedge Q \wedge R)$
DeMorgan's Laws	$\neg(P \vee Q)$	$\equiv$	$(\neg P \wedge \neg Q)$		
	$\neg(P \wedge Q)$	$\equiv$	$(\neg P \vee \neg Q)$		
Distributive Laws	$P \vee (K \wedge Q)$	$\equiv$	$(P \vee K) \wedge (P \vee Q)$		
	$P \wedge (K \vee Q)$	$\equiv$	$(P \wedge K) \vee (P \wedge Q)$		
Subsumption Laws	$P \vee (P \wedge Q)$	$\equiv$	P		
	$P \wedge (P \vee Q)$	$\equiv$	P		
Definition of Implication	$(P \rightarrow Q)$	$\equiv$	$(\neg P \vee Q)$		
Contrapositive	$(P \rightarrow Q)$	$\equiv$	$(\neg Q \rightarrow \neg P)$		
Quantifier Negation	$\neg\exists x : P(x)$	$\equiv$	$\forall x : \neg P(x)$		
	$\neg\forall x : P(x)$	$\equiv$	$\exists x : \neg P(x)$		

A Chart of the Inference Rules of Logic

Note: The symbols P,Q,R,S below can represent complex logical statements. F = "False"

Modus Ponens	P $\underline{P \to Q}$ $\therefore Q$	**Law of Syllogism**	$P \to Q$ $\underline{Q \to R}$ $\therefore P \to R$
Modus Tollens	$P \to Q$ $\underline{\neg Q}$ $\therefore \neg P$	**Rule of Conjunction**	P $\underline{Q}$ $\therefore P \wedge Q$
Rule of Disjunctive Syllogism	$P \vee Q$ $\underline{\neg P}$ $\therefore Q$	**Rule of Contradiction**	$\underline{\neg P \to F}$ $\therefore P$
Rule of Conjunctive Simplification	$\underline{P \wedge Q}$ $\therefore P$	**Rule of Disjunctive Amplification**	$\underline{P}$ $\therefore P \vee Q$
Rule of Conditional Proof	$P \wedge Q$ $\underline{P \to (Q \to R)}$ $\therefore R$	**Rule for Proof by Resolution**	$P \vee Q$ $\underline{\neg P \vee R}$ $\therefore (Q \vee R)$
Rule of the Constructive Dilemma	$P \to Q$ $R \to S$ $\underline{P \vee R}$ $\therefore Q \vee S$	**Rule of the Destructive Dilemma**	$P \to Q$ $R \to S$ $\underline{\neg Q \vee \neg S}$ $\therefore \neg P \vee \neg R$
Rule of Universal Specification (this is valid for any c)	$\underline{\forall x : P(x)}$ $\therefore P(c)$	**Rule of Universal Generalization**	$\underline{P(a) \text{ [a is an arbitrary element]}}$ $\therefore \forall x : P(x)$
Rule of Existential Specification (c is the specific example)	$\underline{\exists x : P(x)}$ $\therefore P(c)$	**Rule of Existential Generalization**	$\underline{P(c) \text{ [c is a specific example]}}$ $\therefore \exists x : P(x)$