C241 Homework 5: Sets

Due Wednesday 10/14

1) Carefully answer the following questions about quantifiers.

a) Two quantifiers of the same type can commute with each other.
So $[\forall x: \ (\forall y: \ P(x,y))] \equiv [\forall y: \ (\forall x: P(x,y))]$,
and $[\exists x: \ (\exists y: \ Q(x,y))] \equiv [\exists y: \ (\exists x: Q(x,y))]$ Come up with specific predicates to fill in for $P(x,y)$ and $Q(x,y)$, and then translate all four statements into english and explain in your own words why they're equivalent.

b) However, quantifiers of different types are *not* commutative.
So $[\forall x: \ (\exists y: \ R(x,y))] \not\equiv [\exists y: \ (\forall x: R(x,y))]$,
and $[\exists x: \ (\forall y: \ S(x,y))] \not\equiv [\forall y: \ (\exists x: S(x,y))]$
Come up with good specific predicates to fill in for $R(x,y)$ and $S(x,y)$, and then translate all four statements into english and explain in your own words why they are *not* equivalent (it will help if you choose predicates which make this point clear).

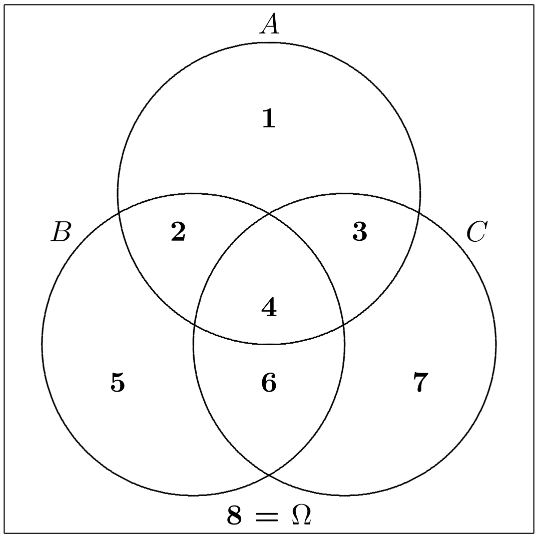
2) List the elements of the following sets.

a) $\{1,2,3,8\} \cap \{2,8\}$
b) $\{1,2,3,8\} \cup \{2,8,5\}$
c) $\{1,2,3,8\} - \{2,8\}$
d) $\{1,2,3,8\} \cap \{5,7\}$
e) $\{1,2,3,8\} \cap \emptyset$
f) $\{1,2,3,8\} \cup \emptyset$
g) power set of $\{2,8\}$

3) Label each of these statemens as either True or False.:

a) $\{2,8\} \subseteq \{1,2,3,8\}$
b) $\{2,8\} \in \{1,3,\{2,8\}\}$
c) $\{2,8\} \subseteq \{1,3,\{2,8\}\}$
d) $\{1,3\} \in \{1,3,\{2,8\}\}$
e) $\emptyset \in \{1,3,2,8,\emptyset\}$
f) $\emptyset \in \{1,3,2,8\}$
g) $\emptyset \subseteq \{1,3,2,8\}$
h) $\{1,2,3,8\} \subseteq \{1,2,3,8\}$
i) $\{1,2,8,32\} \subseteq \{2^n \mid n \in \mathbb{N}\}$
j) $\{\emptyset\} = \emptyset$
k) $\emptyset \subseteq \{\emptyset\}$
l) $\{2n \mid n \in \mathbb{N}\} \subseteq \{2^n \mid n \in \mathbb{N}\}$

Venn Diagram for Three Sets



9 = ∅

4) Write the numbers of the Venn Diagram sections that you would shade to represent the following sets. Use 8 to indicate the *whole* space (or universe), Ω, and 9 to indicate the empty-set, ∅. For instance, set B = 2, 4, 5, 6 while set $A \cap B$ = 2, 4.

a) $B \cap C \cap \overline{A}$

b) $A \cup B$

c) $A \cap B \cap C$

d) $A - B$

e) $A \cap \overline{A}$

f) $A \cup \overline{A}$

g) A

h) $A \cup (A \cap B)$

i) $A \cap (A \cup B)$

5) Determine whether each of the following statements are True or False, and label them accordingly. If the statement is False, correct it so it's True. Remember "$A \Rightarrow B$" does *not* mean the same thing as "$A \equiv B$".

a) $(x \notin B) \equiv \neg(x \in B)$

b) $(B \subseteq A) \equiv \forall x[(x \in B) \rightarrow (x \in A)]$

c) $(B \subseteq A) \equiv \exists x[(x \in B) \rightarrow (x \in A)]$

d) $(B \subseteq A) \Rightarrow \exists x[(x \in B) \rightarrow (x \in A)]$

e) $((B \subseteq A) \vee (A \subseteq B)) \equiv (A = B)$

f) $((B \subseteq A) \wedge (A \subseteq B)) \equiv (A = B)$

g) $\forall x[(x \in B) \rightarrow (x \in A)] \equiv \forall x[\neg(x \in B) \wedge (x \in A)]$

h) $((B \subseteq A) \wedge (x \in B)) \Rightarrow (x \in A)$

i) $(B = \emptyset) \equiv (\exists x[x \in B])$

j) $\neg(\exists x[x \in B]) \equiv \forall x[x \notin B]$

k) $(x \in (A \cap B)) \equiv ((x \in A) \vee (x \in B))$

l) $(x \in (A \cup B)) \equiv ((x \in A) \vee (x \in B))$

m) $(x \notin (A \cup B)) \equiv ((x \notin A) \vee (x \notin B))$

n) $(x \notin (A \cup B)) \Rightarrow ((x \notin A) \vee (x \notin B))$

o) $(x \notin (A \cap B)) \equiv ((x \notin A) \vee (x \notin B))$

p) $(x \notin (A \cap B)) \equiv ((x \in A) \rightarrow (x \notin B))$

q) $[(x \notin (A \cap B)) \wedge (x \in B)] \Rightarrow (x \in A)$

r) $\forall x[(x \in A) \wedge (x \in B)] \Rightarrow \forall x[(x \in A) \vee (x \in B)]$

s) $\forall x[(x \in (A \cap B))] \equiv \forall x[(x \in (A \cup B))]$

6) Prove the following equivalence. Start by replacing the statements $A \subseteq B$ and $A \cap \overline{B} = \emptyset$ with formal logical statements, using $\exists$, $\forall$, and $\in$. (Note that $A = \emptyset$ can be written as $\neg\exists x[x \in A]$), and then show that the statements are logically equivalent.

$$(A \subseteq B) \equiv (A \cap \overline{B} = \emptyset)$$

7) Given the following information, use the inclusion-exclusion principle (page 9) to determine how many total students are enrolled in at least one of the classes: C241, C343, C335. *Show your work.*

- C335 has 15 students enrolled in it (these students may be enrolled in other classes as well)
- C343 has 17 students enrolled in it
- C241 has 20 students enrolled in it
- 5 students are taking both C343 and C335
- 7 students are taking both C241 and C343
- 8 students are taking both C241 and C335
- 4 students are taking all three classes.

<u>Proofs:</u>

8) Easy Proof: If A is a "proper subset" of B, we write $A \subset B$ (instead of $A \subseteq B$, which is the symbol for our regular definition of subset). This means that $(\forall x[(x \in A) \rightarrow (x \in B)]) \wedge (\exists x[(x \in B) \wedge \neg(x \in A)])$. Prove (by means of a clear explanation in plain english) that $\{x \mid (x \in \mathbb{N}) \wedge (x$ is even $)\} \subset \mathbb{N}$

9) Medium Proof: Prove, by giving as clear and precise an english explanation as possible, why: $[(A \subset B) \wedge (B \subset C)] \Rightarrow (A \subset C)$.

10)(*Extra Credit*) Hard Proof: Read section 3.7 and Solved Problems 3.10-3.13 in your book. (a) Prove that the intersection of two countable sets is countable. (b) Prove that the product of two countable sets is countable (see page 23 for a definition of product)