C241 Homework 5: Solutions

1) Carefully answer the following questions about quantifiers.

a) Two quantifiers of the same type can commute with each other.
So $[\forall x: (\forall y: P(x,y))] \equiv [\forall y: (\forall x: P(x,y))]$,
and $[\exists x: (\exists y: Q(x,y))] \equiv [\exists y: (\exists x: Q(x,y))]$ Come up with specific predicates to fill in for $P(x,y)$ and $Q(x,y)$, and then translate all four statements into english and explain in your own words why they're equivalent.

$P(x,y) =$ "x is friends with y" (domain of x disney princesses, domain of y is forest creatures).

Then "It's true that for every princess and every forest creature, the princess is friends with the forest creature" is equivalent to "It's true that for every forest creature and every princess, the princess is friends with the forest creature".

$Q(x,y) =$ "x married y" (domain of x is high school cheerleaders and domain of y is high school football stars)

Then "There's a high school cheerleader and a football star such that the cheerleader married the football star" is equivalent to "There's a football star and a high school cheerleader such that the cheerleader married the football star"

b) However, quantifiers of different types are *not* commutative.
So $[\forall x: (\exists y: R(x,y))] \not\equiv [\exists y: (\forall x: R(x,y))]$,
and $[\exists x: (\forall y: S(x,y))] \not\equiv [\forall y: (\exists x: S(x,y))]$
Come up with good specific predicates to fill in for $R(x,y)$ and $S(x,y)$, and then translate all four statements into english and explain in your own words why they are *not* equivalent (it will help if you choose predicates which make this point clear).

$R(x,y) =$ "x lives in y" (domain of x is people and domain of y is cities)

Then "For every person there is some city that they live in" is *not* equivalent to "There is some city that every person lives in"

$S(x,y) =$ "x has defeated y" (pirates, ninjas)

Then "There's a pirate, such that for all ninja, this pirate has defeated the ninja" (or "There's a pirate who has defeated all ninja") is *not* equivalent to "For each ninja there's a pirate that that has defeated him (or her)".

2) List the elements of the following sets.

a) $\{1,2,3,8\} \cap \{2,8\} = \{2,8\}$

b) $\{1,2,3,8\} \cup \{2,8,5\} = \{1,2,3,8,5\}$

c) $\{1,2,3,8\} - \{2,8\} = \{1,3\}$

d) $\{1,2,3,8\} \cap \{5,7\} = \emptyset$

e) $\{1,2,3,8\} \cap \emptyset = \emptyset$

f) $\{1,2,3,8\} \cup \emptyset = \{1,2,3,8\}$

g) power set of $\{2,8\} = \{\emptyset, \{2\}, \{8\}, \{2,8\}\}$

3) Label each of these statemens as either True or False.:

a) $\{2,8\} \subseteq \{1,2,3,8\}$: T

b) $\{2,8\} \in \{1,3,\{2,8\}\}$: T

c) $\{2,8\} \subseteq \{1,3,\{2,8\}\}$: F

d) $\{1,3\} \in \{1,3,\{2,8\}\}$: F

e) $\emptyset \in \{1,3,2,8,\emptyset\}$: T

f) $\emptyset \in \{1,3,2,8\}$: F

g) $\emptyset \subseteq \{1,3,2,8\}$: T

h) $\{1,2,3,8\} \subseteq \{1,2,3,8\}$: T

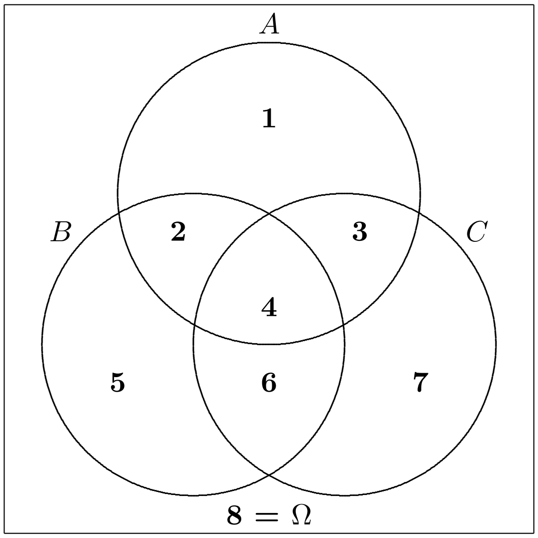
i) $\{1,2,8,32\} \subseteq \{2^n \mid n \in \mathbb{N}\}$

j) $\{\emptyset\} = \emptyset$: F

k) $\emptyset \subseteq \{\emptyset\}$: T

l) $\{2n \mid n \in \mathbb{N}\} \subseteq \{2^n \mid n \in \mathbb{N}\}$: F ($6 \neq 2^n$, for any $n \in \mathbb{N}$).

Venn Diagram for Three Sets



9 $= \emptyset$

4) Write the numbers of the Venn Diagram sections that you would shade to represent the following sets. Use 8 to indicate the *whole* space (or universe), Ω**, and 9 to indicate the empty-set,** $\emptyset$**. For instance, set B = 2, 4, 5, 6 while set** $A \cap B$ **= 2, 4.**

a) $B \cap C \cap \overline{A} = 6$

b) $A \cup B = 1, 2, 3, 4, 5, 6$

c) $A \cap B \cap C = 4$

d) $A - B = 1, 3$

e) $A \cap \overline{A} = 9$

f) $A \cup \overline{A} = 8$

g) $A = 1, 2, 3, 4$

h) $A \cup (A \cap B) = 1, 2, 3, 4$

i) $A \cap (A \cup B) = 1, 2, 3, 4$

5) Determine whether each of the following statements are True or False, and label them accordingly. If the statement is False, correct it so it's True. Remember "$A \Rightarrow B$" does *not* mean the same thing as "$A \Leftrightarrow B$". Inference rules, Laws of Logic, Set Theory definitions, and Quantifier equivalences are all applicable here, so pages 58 and 78, table 2.23, and the definitions in section 3.1-3.2 may all be useful. There may be more than one way to correct the False problems, this is one set of correct answers:

a) $(x \notin B) \Leftrightarrow \neg(x \in B)$

T

b) $(B \subseteq A) \Leftrightarrow \forall x[(x \in B) \rightarrow (x \in A)]$

T

c) $(B \subseteq A) \Leftrightarrow \exists x[(x \in B) \rightarrow (x \in A)]$

F: $(B \subseteq A) \Rightarrow \exists x[(x \in B) \rightarrow (x \in A)]$

d) $(B \subseteq A) \Rightarrow \exists x[(x \in B) \rightarrow (x \in A)]$

T

e) $((B \subseteq A) \vee (A \subseteq B)) \Leftrightarrow (A = B)$

F: $((B \subseteq A) \wedge (A \subseteq B)) \Leftrightarrow (A = B)$

f) $((B \subseteq A) \wedge (A \subseteq B)) \Leftrightarrow (A = B)$

T

g) $\forall x[(x \in B) \rightarrow (x \in A)] \Leftrightarrow \forall x[\neg(x \in B) \wedge (x \in A)]$

F: $\forall x[(x \in B) \rightarrow (x \in A)] \Leftrightarrow \forall x[\neg(x \in B) \vee (x \in A)]$

h) $((B \subseteq A) \wedge (x \in B)) \Rightarrow (x \in A)$

T

i) $(B = \emptyset) \Leftrightarrow (\exists x[x \in B])$
F: $(B = \emptyset) \Leftrightarrow \neg(\exists x[x \in B])$

j) $\neg(\exists x[x \in B]) \Leftrightarrow \forall x[x \notin B]$
T

k) $(x \in (A \cap B)) \Leftrightarrow ((x \in A) \vee (x \in B))$
F: $(x \in (A \cap B)) \Leftrightarrow ((x \in A) \wedge (x \in B))$

l) $(x \in (A \cup B)) \Leftrightarrow ((x \in A) \vee (x \in B))$
T

m) $(x \notin (A \cup B)) \Leftrightarrow ((x \notin A) \vee (x \notin B))$
F: $(x \notin (A \cup B)) \Leftrightarrow ((x \notin A) \wedge (x \notin B))$

n) $(x \notin (A \cup B)) \Rightarrow ((x \notin A) \vee (x \notin B))$
T

o) $(x \notin (A \cap B)) \Leftrightarrow ((x \notin A) \vee (x \notin B))$
T

p) $(x \notin (A \cap B)) \Leftrightarrow ((x \in A) \rightarrow (x \notin B))$
T

q) $[(x \notin (A \cap B)) \wedge (x \in B)] \Rightarrow (x \in A)$
F: $[(x \notin (A \cap B)) \wedge (x \in B)] \Rightarrow (x \notin A)$

r) $\forall x[(x \in A) \wedge (x \in B)] \Rightarrow \forall x[(x \in A) \vee (x \in B)]$
T

s) $\forall x[(x \in (A \cap B))] \Leftrightarrow \forall x[(x \in (A \cup B))]$
F: $\forall x[(x \in (A \cap B))] \Rightarrow \forall x[(x \in (A \cup B))]$

6) Do problem 6 (b) in section 3.2 of your book. Start by replacing the statements $A \subseteq B$ and $A \cap \overline{B} = \emptyset$ with formal logical statements, using $\exists$, $\forall$, and $\in$. (Note that $A = \emptyset$ can be written as $\neg\exists x[x \in A]$), and then show that the statements are logically equivalent.

$A \subseteq B$ can be written as $\forall x[(x \in A) \to (x \in B)]$

$A \cap \overline{B} = \emptyset$ can be written as $\neg\exists x[x \in (A \cap \overline{B})]$

$\neg\exists x[x \in (A \cap \overline{B})] \Leftrightarrow$	
$\forall x \neg[x \in (A \cap \overline{B})] \Leftrightarrow$	Quantifier negation rules
$\forall x \neg[(x \in A) \wedge (x \in \overline{B})] \Leftrightarrow$	Definition of Intersection
$\forall x \neg[(x \in A) \wedge \neg(x \in B)] \Leftrightarrow$	Definition of Complement
$\forall x[\neg(x \in A) \vee \neg\neg(x \in B)] \Leftrightarrow$	DeMorgan's
$\forall x[\neg(x \in A) \vee (x \in B)] \Leftrightarrow$	Double Negation
$\forall x[(x \in A) \to (x \in B)] \Leftrightarrow$	$(\neg p \vee q) \Leftrightarrow (p \to q)$
$A \subseteq B$	Definition of subset

7) Given the following information, use the inclusion-exclusion principle (page 9) to determine how many total students are enrolled in at least one of the classes: C241, C343, C335. *Show your work.*

- C335 has 15 students enrolled in it (these students may be enrolled in other classes as well)
- C343 has 17 students enrolled in it
- C241 has 20 students enrolled in it
- 5 students are taking both C343 and C335
- 7 students are taking both C241 and C343
- 8 students are taking both C241 and C335
- 4 students are taking all three classes.

The inclusion-exclusion principle states that: $|(A \cup B \cup C)| = |A| + |B| + |C| - |(A \cap B)| - |(A \cap C)| - |(B \cap C)| + |(A \cap B \cap C)|$.

So, if A = {students in C241}, B = {students in C343 }, C = {students in C335}, then number of students in the students union of these sets is : $|(A \cup B \cup C)| = 20 + 17 + 15 - 7 - 8 - 5 + 4 = 36$

Proofs:

8) Easy Proof: If A is a "proper subset" of B, we write $A \subset B$ (instead of $A \subseteq B$, which is the symbol for our regular definition of subset). This means that $(\forall x[(x \in A) \rightarrow (x \in B)]) \wedge (\exists x[(x \in B) \wedge \neg(x \in A)])$. Prove (by means of a clear explanation in plain english) that $\{x \mid (x \in \mathbb{N}) \wedge (x$ is even $)\} \subset \mathbb{N}$

So the definition $(\forall x[(x \in A) \rightarrow (x \in B)]) \wedge (\exists x[(x \in B) \wedge \neg(x \in A)])$ means that every element of A is also in B, but there's at least one element in B that's not in A. So we have $A \subseteq B$, but $A \neq B$. Just using the definitions, we can prove that $\{x \mid (x \in \mathbb{N}) \wedge (x \text{ is even })\} \subset \mathbb{N}$ relatively simply: The first set is the even natural numbers, and the second set is the natural numbers (all of them). Every even natural number is a natural number (so every element of the first set is an element of the second; the first set is a subset of the second). But there's at least one natural number that's not an even natural number: the number 3 for instance. So the two sets are not equal. Thus the first set is a proper subset of the second.

9) Medium Proof: Prove, by giving as clear and precise an english explanation as possible, why: $[(A \subset B) \wedge (B \subset C)] \Rightarrow (A \subset C)$.

Again, the best way to do this is just by following the definition. To show that $(A \subset C)$, we need to show that $(\forall x[(x \in A) \rightarrow (x \in C)]) \wedge (\exists x[(x \in C) \wedge \neg(x \in A)])$. We know that $(A \subset B)$ and $(B \subset C)$. So every element $x \in A$ is also in B, and if $x \in B$ then it's also in C. So we've got $(\forall x[(x \in A) \rightarrow (x \in C)])$. We also know that C has some additional element that's not in B (since B is a proper subset of C), and if this element isn't in B it also can't be in A (since $A \subset B$). So we've got $(\exists x[(x \in C) \wedge \neg(x \in A)])$. And thus, we've proved that $A \subset C$.

10)(*Extra Credit*) Hard Proof: Read section 3.7 and Solved Problems 3.10-3.13 in your book. (a) Prove that the intersection of two countable sets is countable. (b) Prove that the product of two countable sets is countable (see page 23 for a definition of product)

Here's the solution to these problems and one more:

If A and B are infinite countable sets, prove that the following are also countable sets. Remember that a countable set is any set where you

can find an injective function which maps the elements of the set to the natural numbers. Also, any set whose elements can all be written out in sequence will be countable, since there's an easy injective function which maps each element to the number that represents that element's spot in the sequence. You can assume that if a set A is countable, then its elements can be written out: $A = \{a_1, a_2, a_3...\}$

a) $A \cup B$

If A is countable, you can list out its elements in sequence: $A = (a_1, a_2, a_3...)$. And you can do the same thing with B: $B = (b_1, b_2, b_3...)$. To prove that $A \cup B$ is countable, we just need to find a sequence that will list out all the elements in A and B. For instance: $A \cup B = (a_1, b_1, a_2, b_2, a_3, b_3...)$. (Note that the sequence $(a_1, a_2, a_3...b_1, b_2...)$ won't work because if there's an infinite number of elements in A, you'll never get to list the elements in B.)

b) $A \cap B$

Remember that $A \cap B \subseteq A$. Since A is countable, there's an injective function from A to the integers. And because $A \cap B \subseteq A$, that same function will map $A \cap B$ to the integers too. So $A \cap B$ is also countable.

c) $A \times B$ (you can use a dove-tailing argument here)

Above we said you could list out the elements of A in sequence: $A = (a_1, a_2, a_3...)$, and the same thing with B: $B = (b_1, b_2, b_3...)$. So, we can write $A \times B$ as:

$$\begin{array}{llllll} \{ & (a_1, b_1), & (a_1, b_2), & (a_1, b_3), & (a_1, b_4), & ... \\ & (a_2, b_1), & (a_2, b_2), & (a_2, b_3), & ... & \\ & (a_3, b_1), & (a_3, b_2), & ... & & \\ & (a_4, b_1), & ... & & & \\ & ... \} & & & & \end{array}$$

Then we can list out all the elements of $A \times B$ in a diagonal sequence that's known as 'dove-tailing': $A \times B = ((a_1, b_1), (a_1, b_2), (a_2, b_1), (a_3, b_1), (a_2, b_2), (a_1, b_3), (a_1, b_4), (a_2, b_3), (a_3, b_2), (a_4, b_1), ...)$.