C241 Homework 6

Due Wednesday, 10/21/09

1) List the elements in the following sets.

A = **{1, 2, 3}** **B = $\emptyset$** **C={a, b}**

a) A $\times$ C =

b) C $\times$ A =

c) C^3 = C $\times$ C $\times$ C =

d) (C^2) $\times$ A = (C $\times$ C) $\times$ A

e) (C^3) $\times$ B =

2) Match each formal definition with its informal english equivalent.

a) $\{a \in A | \exists b \in S \text{ such that } (a,b) \in F\}$

b) $R \subseteq A \times B$

c) $F : A \rightarrow B$ is a function, and $\forall b \in B$, $\exists a \in A$ such that $(a,b) \in F$

d) $\{(b,a)|(a,b) \in F\}$

e) $F \subseteq A \times B$ and $\forall a \in A, \exists$ *exactly one* $b \in B$ such that $(a,b) \in F$

f) $\{b \in B | \exists a \in S \text{ such that } (a,b) \in F\}$

g) $F : A \rightarrow B$ is a function, and $\forall b \in B$, $\exists$ *at most* one $a \in A$ such that $(a,b) \in F$

h) $\{(a,c) \mid \exists b \in B \text{ such that } (a,b) \in F \text{ and } (b,c) \in G\}$

i) A ***relation*** $R : A \rightarrow B$ is a set of ordered pairs, where the first element in the pair comes from the set A, and the second element in the pair comes from the set B. We say the first element in each pair is "mapped" to the second element.

ii) A ***function*** $F : A \rightarrow B$ is a relation in which every element in A is paired with *exactly one* element in B. This means that each element in A is paired with an element in B, and no element in A is paired with more than one element in B.

iii) The ***image*** of a set $S \subseteq A$ under a function $F : A \rightarrow B$ is the set of all elements in B that F pairs with the elements of A that are in S.

iv) The ***pre-image*** of a set $S \subseteq B$ under a function $F : A \rightarrow B$ is the set of all elements in A that F pairs with the elements of B that are in S.

v) An ***injective function*** $F : A \rightarrow B$ is a function in which no two elements of A are paired with the same element in B.

vi) A ***surjectve function*** $F : A \rightarrow B$ is a function in which every element of B is paired with some element in A.

vii) F^{-1}, the ***inverse*** of a function $F : A \rightarrow B$, is the relation consisting of all the pairs in F flipped so that the second element is first, and the first element is second. The inverse of a function isn't always a function itself.

viii) $(G \circ F) : A \rightarrow C$, the ***composition*** of two functions $F : A \rightarrow B$ and $G : B \rightarrow C$, is the set of all pairs (a, c) where (a, b) is a pair in F and (b, c) is a pair in G.

3) Using the sets $A = \{1, 2, 3, 4\}, B = \{a, b, c, d\}$ and $C = \{x, y, z\}$, give examples for each of the terms that were defined in problem 3 above.

a) relation

b) function

c) image (use your F from part b)

d) preimage (use your F from part b)

e) injective function

f) surjective function

g) bijective function (a function that's both injective and surjective)

h) inverse (use your F from part g)

i) function composition

4) Let A = {1, 2} and B = {2, 3, 4}. Which of the following relations from A to B are functions?

a) { (1, 3), (2, 4) }   c) { (1, 3), (1, 3) }   e) { (1, 3), (2, 5) }
b) { (1, 3), (1, 4) }   d) { (2, 2), (1, 4) }

5) Given a (total) function F = {(1, 2), (2, 3)}, which of the following are a *valid* domain and range for F?

a) F: $\mathbb{N} \rightarrow \mathbb{N}$
b) F: $\{1, 2\} \rightarrow \mathbb{N}$
c) F: $\{1, 2, 3\} \rightarrow \{1, 2, 3\}$
d) F: $\{(1, 2)\} \rightarrow \{(2, 3)\}$
e) F: $\{1, 2, 3\} \rightarrow \{2, 3\}$

6) Label the following functions as: injective, surjective, both (bijective), or neither.

a) $F : \{1, 2, 3\} \rightarrow \{a, b\}, F = \{(1, a), (2, b), (3, b)\}$

b) $F : \{1, 2\} \rightarrow \{a, b, c\}, F = \{(1, a), (2, b)\}$

c) $F : \{1, 2, 3\} \rightarrow \{a, b, c\}, F = \{(1, a), (3, b), (2, c)\}$

d) $F : \{1, 2, 3\} \rightarrow \{a, b, c\}, F = \{(1, b), (2, b), (3, a)\}$

e) $F : \{1, 2\} \rightarrow \{a, b, c\}, F = \{(1, a), (2, a)\}$

7) One way of proving that a function $F : A \rightarrow B$ is injective is to show that if both $(a_1, b) \in F$ and $(a_2, b) \in F$, then $a_1 = a_2$.

For example, given $F : \mathbb{R} \rightarrow \mathbb{R}$, $F(x) = x^3$, we can prove that F is injective as follows:

If $F(a_1) = b$ and $F(a_2) = b$, then $F(a_1) = F(a_2)$,
which means that $(a_1)^3 = (a_2)^3$, so $\sqrt[3]{(a_1)^3} = \sqrt[3]{(a_2)^3}$,
which reduces to: $a_1 = a_2$.

For each of the functions below, determine whether the function is injective or not. If the function is injective, prove that it is, using a short mathematical proof in the style above. If the function is not injective, give an example of a_1, a_2 such that $F(a_1) = F(a_2)$, but $a_1 \neq a_2$

a) $F : \mathbb{R} \rightarrow \mathbb{R}$, $F(y) = 17 \times y$

b) $F : \mathbb{R} \rightarrow \mathbb{R}$, $F(x) = x^2$

c) $F : \mathbb{N} \rightarrow \mathbb{N}$, $F(w) = 2^w$

d) $F : \{a\}^+ \rightarrow \mathbb{N}$, $F(s) = length(s)$ (where $\{a\}^+ = \{a, aa, aaa, aaaa, aaaaa, aaaaaa.....\}$ is the set of all strings of a's.)

8) Label the following claims as True or False. If you think a claim is true, *use the definitions to explain clearly and precisely why it's true.* If you think it's false, *give an example that shows that it is false.* (note, when you're thinking about these problems, it may help to draw pictures of the functions such as the ones in figure 2.3 on page 26. However, to get full credit, your proof needs to consist of more than just a picture.) The first and third have been done for you.

Example I) If there is a surjective function $F : A \to B$, then $|B| \leq |A|$.
Proof by contradiction: Let's look at what would happen if $|A| < |B|$. Because F is a function, each element in A is paired with exactly one element in B... which means that the set of elements in B that are paired with elements in A has size at most $|A|$ (this set is the "range" of F). So if $|B| > |A|$, there are $(|B| - |A|)$ elements left over in B which aren't paired with any element of A. But this contradicts the fact that F is surjective. Since we've found a contradiction, it must be the fact that $|A| < |B|$ is false, ie that $|B| \leq |A|$.

a) If there is an injective function $G : A \to B$, then $|B| \geq |A|$.

Example II) If there is a bijective function $H : A \to B$, then $|B| = |A|$.
If H is a bijection, that means that it's both surjective and injective. We know from Example I that this means $|B| \leq |A|$, and we'll know from (a) (after you do it) that $|B| \geq |A|$. The only way these can both be true is if $|A| = |B|$.

b) If $F : A \to B$ is a surjective function, but it is *not* injective, then its inverse F^{-1} is also a function.

c) If $F : A \to B$ is a surjection, and there is another function $G : B \to A$ which is also a surjection, then you can find a third function $H : A \to B$ such that H is a bijection.

d) If $|A| < |B|$ then there can be a surjective function $F : A \to B$

e) If $F : A \to B$ is an injective function, and $G : B \to C$ is also an injective function, then the composition $(G \circ F) : A \to C$ is injective.

f) If $S \subseteq A$ and $S \neq \emptyset$, and $F : A \to B$ is a total function, then $|image(S)| \geq 1$.

<u>Proofs:</u>

A set A is called "countable" if it's possible to make an injective function $F : A \to \mathbb{N}$. Note: countable *isn't* the same thing as finite. For instance, $\mathbb{N}$ itself is countable, since there's obviously an injective function $F : \mathbb{N} \to \mathbb{N}$ (just map each number to itself). You can think of this as a generalization of what you actually do when you count a group of things outloud, using your fingers "1, 2, 3, 4...". If you had infinite fingers to count on.

8) Easy Proof: Prove that the set $\{a, aa, aaa, aaaa, aaaaa, ...\}$ (the set of all strings of a's) is countable.

9) Medium Proof: Prove that any finite set is countable.

10)(*Extra Credit*) Hard Proof: Prove that the set of rational numbers: $\{\frac{m}{n} \mid m, n \in \mathbb{N}\}$, is countable.