

C241 Homework 6

Due Wednesday, 10/21/09

1) List the elements in the following sets.

$$A = \{1, 2, 3\} \quad B = \emptyset \quad C = \{a, b\}$$

$$\text{a) } A \times C = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$\text{b) } C \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

$$\text{c) } C^3 = C \times C \times C = \{(a, a, a), (a, a, b), (a, b, a), (a, b, b), (b, a, a), (b, a, b), (b, b, a), (b, b, b)\}$$

$$\text{d) } (C^2) \times A = (C \times C) \times A = \{((a, a), 1), ((a, a), 2), ((a, b), 1), ((a, b), 2), ((b, a), 1), ((b, a), 2), ((b, b), 1), ((b, b), 2)\}$$

$$\text{e) } (C^3) \times B = \emptyset$$

2) Match each formal definition with its informal english equivalent.

i) A **relation** $R : A \rightarrow B$ is a set of ordered pairs, where the first element in the pair comes from the set A, and the second element in the pair comes from the set B. We say the first element in each pair is "mapped" to the second element.

b) $R \subseteq A \times B$

ii) A **function** $F : A \rightarrow B$ is a relation in which every element in A is paired with *exactly one* element in B. This means that each element in A is paired with an element in B, and no element in A is paired with more than one element in B.

e) $F \subseteq A \times B$ and $\forall a \in A, \exists$ *exactly one* $b \in B$ such that $(a, b) \in F$

iii) The **image** of a set $S \subseteq A$ under a function $F : A \rightarrow B$ is the set of all elements in B that F pairs with the elements of A that are in S.

f) $\{b \in B | \exists a \in S \text{ such that } (a, b) \in F\}$

iv) The **pre-image** of a set $S \subseteq B$ under a function $F : A \rightarrow B$ is the set of all elements in A that F pairs with the elements of B that are in S.

a) $\{a \in A | \exists b \in S \text{ such that } (a, b) \in F\}$

v) An **injective function** $F : A \rightarrow B$ is a function in which no two elements of A are paired with the same element in B.

g) $F : A \rightarrow B$ is a function, and $\forall b \in B, \exists$ *at most one* $a \in A$ such that $(a, b) \in F$

vi) A **surjective function** $F : A \rightarrow B$ is a function in which every element of B is paired with some element in A.

c) $F : A \rightarrow B$ is a function, and $\forall b \in B, \exists a \in A$ such that $(a, b) \in F$

vii) F^{-1} , the **inverse** of a function $F : A \rightarrow B$, is the relation consisting of all the pairs in F flipped so that the second element is first, and the first element is second. The inverse of a function isn't always a function itself.

d) $\{(b, a) | (a, b) \in F\}$

viii) $(G \circ F) : A \rightarrow C$, the **composition** of two functions $F : A \rightarrow B$ and $G : B \rightarrow C$, is the set of all pairs (a, c) where (a, b) is a pair in F and (b, c) is a pair in G .

h) $\{(a, c) \mid \exists b \in B \text{ such that } (a, b) \in F \text{ and } (b, c) \in G\}$

3) Using the sets $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$, $C = \{x, y, z\}$, give examples for each of the terms that were defined in problem 2 above.

a) relation $\{(1, a), (3, c), (3, d)\}$

b) function $\{(1, b), (3, d), (2, c), (4, c)\}$

c) image (use your F from part b) $S = \{2, 4\}$ Image of $S = \{c\}$

d) preimage (use your F from part b) $T = \{b, c\}$ Pre-image of $T = \{1, 2\}$

e) injective function $\{(1, a), (2, b), (3, d), (4, c)\}$

f) surjective function $\{(1, a), (2, b), (3, d), (4, c)\}$

g) bijective function (a function that's both injective and surjective)
 $\{(1, a), (2, b), (3, d), (4, c)\}$

h) inverse (use your F from part g) $\{(a, 1), (b, 2), (d, 3), (c, 4)\}$

i) function composition

Using the $F : A \rightarrow B$ that was defined above, and $G : B \rightarrow C$ where $G = \{(1, x), (2, x), (3, y), (4, y)\}$, we get $G \circ F = \{(a, x), (b, x), (c, y), (d, y)\}$

4) Let $A = \{1, 2\}$ and $B = \{2, 3, 4\}$. Which of the following relations from A to B are functions?

a) $\{(1, 3), (2, 4)\}$ × c) $\{(1, 3), (1, 3)\}$ × e) $\{(1, 3), (2, 5)\}$
 × b) $\{(1, 3), (1, 4)\}$ d) $\{(2, 2), (1, 4)\}$

(On e) note that $5 \notin B$)

5) Given a function $F = \{(1, 2), (2, 3)\}$, which of the following are a *valid* domain and range for F ?

There was a mistake in this question. It should have used "co-domain" rather than "range". As the question was written, none of these are a valid domain and range for the function. If it had said co-domain, b) is the only correct answer. (note that on (d) $\{(1, 2)\} \neq \{1, 2\}$, which is the proper domain for F)

- a) $F: \mathbb{N} \rightarrow \mathbb{N}$ d) $F: \{(1, 2)\} \rightarrow \{(2, 3)\}$
 b) $F: \{1, 2\} \rightarrow \mathbb{N}$ e) $F: \{1, 2, 3\} \rightarrow \{2, 3\}$
 c) $F: \{1, 2, 3\} \rightarrow \{1, 2, 3\}$

6) Label the following functions as: injective, surjective, both (bijective), or neither.

- a) $F: \{1, 2, 3\} \rightarrow \{a, b\}, F = \{(1, a), (2, b), (3, b)\}$ surjective
 b) $F: \{1, 2\} \rightarrow \{a, b, c\}, F = \{(1, a), (2, b)\}$ injective
 c) $F: \{1, 2, 3\} \rightarrow \{a, b, c\}, F = \{(1, a), (3, b), (2, c)\}$ bijective
 d) $F: \{1, 2, 3\} \rightarrow \{a, b, c\}, F = \{(1, b), (2, b), (3, a)\}$ neither
 e) $F: \{1, 2\} \rightarrow \{a, b, c\}, F = \{(1, a), (2, a)\}$ neither

7) One way of proving that a function $F: A \rightarrow B$ is injective is to show that if both $(a_1, b) \in F$ and $(a_2, b) \in F$, then $a_1 = a_2$.

For example, given $F: \mathbb{R} \rightarrow \mathbb{R}, F(x) = x^3$, we can prove that F is injective as follows:

If $F(a_1) = b$ and $F(a_2) = b$, then $F(a_1) = F(a_2)$,
 which means that $(a_1)^3 = (a_2)^3$, so $\sqrt[3]{(a_1)^3} = \sqrt[3]{(a_2)^3}$,
 which reduces to: $a_1 = a_2$.

For each of the functions below, determine whether the function is injective or not. If the function is injective, prove that it is, using a short mathematical proof in the style above. If the function is not injective, give an example of a_1, a_2 such that $F(a_1) = F(a_2)$, but $a_1 \neq a_2$

- a) $F: \mathbb{R} \rightarrow \mathbb{R}, F(y) = 17 \times y$

Injective: If $F(x) = F(y)$, then $17x = 17y$, and $x = y$.

b) $F : \mathbb{R} \rightarrow \mathbb{R}$, $F(x) = x^2$ Not Injective: $F(-1) = (-1)^2 = 1$, and $F(1) = 1^2 = 1$, so $F(-1) = F(1)$.

But $-1 \neq 1$

c) $F : \mathbb{N} \rightarrow \mathbb{N}$, $F(w) = 2^w$

Injective: If $F(x) = F(y)$, then $2^x = 2^y$, and $\log_2(2^x) = \log_2(2^y)$

Which reduces to $x = y$

d) $F : \{a\}^+ \rightarrow \mathbb{N}$, $F(s) = \text{length}(s)$ (where $\{a\}^+ = \{a, aa, aaa, aaaa, aaaaa, aaaaaa, \dots\}$ is the set of all strings of a 's.)

Injective: If $F(x) = F(y)$, then $\text{length}(x) = \text{length}(y)$, and since x and y are both just strings of a 's, this means $x = y$.

8) Label the following claims as True or False. If you think a claim is true, use the definitions to explain clearly and precisely why it's true. If you think it's false, give an example that shows that it is false. (note, when you're thinking about these problems, it may help to draw pictures of the functions such as the ones in figure 2.3 on page 26. However, to get full credit, your proof needs to consist of more than just a picture.) The first and third have been done for you.

Example I) If there is a surjective function $F : A \rightarrow B$, then $|B| \leq |A|$.

Proof by contradiction: Let's look at what would happen if $|A| < |B|$. Because F is a function, each element in A is paired with exactly one element in B ... which means that the set of elements in B that are paired with elements in A has size at most $|A|$ (this set is the "range" of F). So if $|B| > |A|$, there are $(|B| - |A|)$ elements left over in B which aren't paired with any element of A . But this contradicts the fact that F is surjective. Since we've found a contradiction, it must be the fact that $|A| < |B|$ is false, ie that $|B| \leq |A|$.

a) If there is an injective function $G : A \rightarrow B$, then $|B| \geq |A|$.

Proof by contradiction: Let's look at what would happen if $|B| < |A|$. Because F is a function, each element in A is paired with exactly one element in B and, since

it's injective, no two items in A are paired with the same item in B ... which means that range of F has exactly $|A|$ elements in it (there's one item in B for every item in A). But if $|A| > |B|$, then B isn't large enough for the range of F to have size $|A|$. This is a contradiction. So, it must be the fact that $|A| < |B|$ is false, ie that $|B| \leq |A|$.

Example II) If there is a bijective function $H : A \rightarrow B$, then $|B| = |A|$.

If H is a bijection, that means that it's both surjective and injective. We know from Example I that this means $|B| \leq |A|$, and we'll know from (a) (after you do it) that $|B| \geq |A|$. The only way these can both be true is if $|A| = |B|$.

b) If $F : A \rightarrow B$ is a surjective function, but it is *not* injective, then its inverse F^{-1} is also a function.

This is false. For a counter-example, consider $A = \{a, b, c\}$, $B = \{1\}$, $F = \{(a, 1), (b, 1), (c, 1)\}$. F is surjective because every element of B is paired with some element in A . But $F^{-1} = \{(1, a), (1, b), (1, c)\}$ clearly isn't a function.

c) If $F : A \rightarrow B$ is a surjection, and there is another function $G : B \rightarrow A$ which is also a surjection, then you can find a third function $H : A \rightarrow B$ such that H is a bijection.

This is true. We can use the result from example one here. Since F is a surjection, we know that $|B| \leq |A|$, and since G is a surjection, we know that $|B| \geq |A|$. The only way these can both be true is if $|B| = |A|$. In that case, we can define a bijection H by picking an order to write the elements of A in, then picking an order for the elements of B , and then having H pair the first item in A with the first item in B , pair the second item in A with the second one in B , and so on. Since A and B are the same size, this will make sure all the elements in B are paired with an element in A (surjective), and we won't run out of items in B so we'll successfully pair each item in A with a different item in B (injective). So H will be a bijection. Since we proved that it's possible for a bijection H to exist by describing a method for building one, this is called a "proof by construction".

d) If $|A| < |B|$ then there can be a surjective function $F : A \rightarrow B$

No, this violates what we proved in Example 1.

e) If $F : A \rightarrow B$ is an injective function, and $G : B \rightarrow C$ is also an injective function, then the composition $(G \circ F) : A \rightarrow C$ is injective.

This is also true. Since G is injective, G won't pair any two elements of B with the same element of C . This means that if you take some element $c \in C$ there will be at most one element $b \in B$ such that the pair (b, c) is in G . If there is such a b paired with that c , then F won't pair more than one element of A with that b , because F is also injective (and so F won't pair any two elements of A with the same $b \in B$). So for any $c \in C$, there's at most one $(b, c) \in G$ and at most one $(a, b) \in F$, which means that for any $c \in C$ there's at most one $(a, c) \in F \circ G$. Thus $F \circ G$ is injective.

f) If $S \subseteq A$ and $S \neq \emptyset$, and $F : A \rightarrow B$ is a total function, then $|\text{image}(S)| \geq 1$.

If F is a total function then F maps every element of A to some element of B . And that means that every element in $S \subset A$ is paired with some element in B . They might all be paired with the same $b \in B$ (if the function isn't injective), or there might only be one element in S , but in either case, if $S \neq \emptyset$ the smallest $|\text{image}(S)|$ can possibly be is 1.

Proofs:

A set A is called "countable" if it's possible to make an injective function $F : A \rightarrow \mathbb{N}$. Note: countable *isn't* the same thing as finite. For instance, \mathbb{N} itself is countable, since there's obviously an injective function $F : \mathbb{N} \rightarrow \mathbb{N}$ (just map each number to itself). You can think of this as a generalization of what you actually do when you count a group of things outloud, using your fingers "1, 2, 3, 4...". If you had infinite fingers to count on.

8) Easy Proof: Prove that the set $\{a, aa, aaa, aaaa, aaaaa, \dots\}$ (the set of all strings of a 's) is countable.

This is simple. We showed in 7d that $\text{length}(s)$ is an injective function from this set to the natural numbers. So by the definition, this is countable. Intuitively, we can list out all the strings of a 's in order, by their length starting with the shortest and going up, so if we could count forever we'd be able to count all of them in that order. For some sets, such as the real numbers, this isn't possible.

9) Medium Proof: Prove that any finite set is countable.

If a set is finite, say it has size n , then we can just pick some order (*any* order) to list out it's elements and we can make an injective function from the set to \mathbb{N} by pairing the first item in the list with 0, the second item with 1, and so on up to the n th item, which will be paired with n . Alternatively, you could note that whether or not a set is countable is a property of its size (essentially uncountable sets are a 'larger' size of infinite than countable ones, although at that point the proper term is 'cardinality' rather than 'size'). Any finite set must be smaller than \mathbb{N} , so since \mathbb{N} is countable all finite sets are countable too.

10)(Extra Credit) Hard Proof: Prove that the set of rational numbers: $\{\frac{m}{n} \mid m, n \in \mathbb{N}\}$, is countable.

For this, I'm going to assume that you can look up the solutions to the bonus problem on assignment 5 online, and I'll point out that writing $\frac{m}{n}$ is not substantially different than writing (m, n) .

Below is the diagonalization argument which proves that the real numbers between 0 and 1 are uncountable. If you haven't read the solutions to the bonus problem on assignment 5, it might be good to review them before reading this so you're more familiar with the basic techniques.

We'll do this as a proof by contradiction... which means that we'll assume that the real numbers between 0 and 1 *are* countable, and then we'll show that that's ridiculous (because it implies something that can't be true—a contradiction).

So, let's assume that the real numbers between 0 and 1 are countable. Then we can list them all out in sequence. We don't know what this sequence would be, but we know it would exist and we can use variables to represent it. Let's say that X_i is the i^{th} number in this sequence. We'll call the j^{th} digit of this number: $x_{i,j}$, (so if the first number in the sequence were 0.1415926... then $x_{1,4}$ is the fourth digit of the first number in the sequence, so $x_{1,4}$ would be 5). Also, we'll take all short decimal numbers like 0.5 and add an infinite number of 0's to the end, so we can write the whole sequence as:

$$X_1 = 0.x_{1,1} \ x_{1,2} \ x_{1,3} \ x_{1,4} \ x_{1,5} \dots$$

$$X_2 = 0.x_{2,1} \ x_{2,2} \ x_{2,3} \ x_{2,4} \ x_{2,5} \dots$$

$$X_3 = 0.x_{3,1} \ x_{3,2} \ x_{3,3} \ x_{3,4} \ x_{3,5} \dots$$

$$X_4 = 0.x_{4,1} \ x_{4,2} \ x_{4,3} \ x_{4,4} \ x_{4,5} \dots$$

$$X_5 = 0.x_{5,1} \ x_{5,2} \ x_{5,3} \ x_{5,4} \ x_{5,5} \dots$$

...

Now I can find a real number that's between 0 and 1, but I know it *isn't* anywhere in this sequence. I'm going to look at the "diagonal" elements of the sequence: $x_{1,1}$, $x_{2,2}$, $x_{3,3}$, $x_{4,4}$, $x_{5,5}$... and if $x_{1,1}$ isn't 3, then the first digit of my number will be 3. If $x_{1,1}$ *is* 3, then the first digit of my number will be 7. So, I at least know that my number won't be X_1 , because I know the first digit will be different. Then I'm going to do the same with with X_2 ... if $x_{2,2}$ isn't 3, then the second digit of my number will be 3. If it *is* 3, the second digit of my number will be 7. So my number won't be X_2 either, because the second digit will be different. If I keep doing this, my number won't be any of the X_i , which means my number isn't in the sequence. And the sequence was supposed to list out *all* real numbers between 0 and 1.... so we've found our contradiction, and the real numbers must not be countable.