C241 Homework 7

Due Wednesday, 10/28/09

1) Match the formal definitions for the following properties of relations (and the graphs that represent them) with their informal english equivalents:

a) $\forall (x, y) \in R$ such that $x \neq y[(x, y) \in R \Rightarrow (y, x) \in R]$

b) Given that R is an equivalence relation: $\{y \in A | (x, y) \in R\}$

c) $\forall x \in A[x \in A \Rightarrow (x, x) \in R]$

d) R is transitive, symmetric, and reflexive.

e) A partition of S into distinct sets: $S_1, S_2, ...S_n$ such that if $x \in S_i$ and $(x, y) \in R$ then $y \in S_i$.

f) R is reflexive, transitive and anti-symmetric.

g) $\forall (x, y), (y, z) \in R[(x, y), (y, z) \in R \Rightarrow (x, z) \in R]$

h) $\forall (x, y) \in R$ such that $x \neq y[(x, y) \in R \Rightarrow (y, x) \notin R]$

i) R is reflexive, transitive and anti-symmetric,
and $\forall x, y \in A[(x, y) \in R$ or $(y, x) \in R]$

i) A relation $R \subseteq A \times A$ is *reflexive* if for every element x in A, there's a pair (x, x) in R. When you draw R as a directed graph, this means that each point in the graph has an arrow pointing to itself.

ii) A relation $R \subseteq A \times A$ is *symmetric* if, for every pair (x, y) that appears in R (where x ≠ y), the reverse of that pair, (y, x), also appears in R. When you draw R as a directed graph, this means that if there's an arrow from x to y, then there is also an arrow from y to x.

iii) A relation $R \subseteq A \times A$ is *anti-symmetric* if, for every pair (x, y) that appears in R (where x ≠ y), the reverse of that pair, (y, x), does **not** appear in R. When you draw R as a directed graph, this means that if there's an arrow from x to y, then there is **not** an arrow from y to x.

iv) A relation $R \subseteq A \times A$ is *transitive* if, whenever you have two pairs such that the last element of one pair is the first element of the other: (x, y) (y, z), you also have a pair: (x, z). When you draw R as a directed graph, this means that whenever you can get from one point to another by following a sequence of arrows, there is also a single arrow that connects the first point to the second.

v) A relation $R \subseteq A \times A$ is an *equivalence relation* if it is reflexive, symmetric and transitive. When you draw R as a directed graph, this means that the graph will partition the points in A into disjoint clumps of fully connected points (these clumps are referred to as equivalence classes).

vi) The *equivalence class* of an element $x \in A$ under an equivalence relation R, is the set of all elements in A which R pairs with x. This is written as $[x]_R$. Since equivalence relations are reflexive, the equivalence class of x will always include x. When you draw R as a directed graph, all the elements which are clumped with x are in the same equivalence class as x.

vii)The set of equivalences classes that an equivalence relation R breaks A into is written as A/R. This will be a partition of A, which means that it will be a set of subsets of A, and each element of A will appear in exactly one of the subsets.

viii) A relation $R \subseteq A \times A$ is a *partial ordering* if it is reflexive, anti-symmetric and transitive. $\subseteq$ is a partial ordering over any set A.

ix) A relation $R \subseteq A \times A$ is a *total ordering* if it is reflexive, anti-symmetric and transitive, and if for any two elements x, y in A, x and y are related to each other by R: either $(x, y) \in R$ or $(y, x) \in R$. "$\geq$" is a total ordering over the natural numbers.

2) Given A = {a, b}, draw the graphs for *every* possible relation R such that $R \subseteq A \times A$. Label each graph with the properties it has: reflexive, symmetric, anti-symmetric, and/or transitive. Make sure you pay careful attention to the definitions above, this can be tricky.

3) Give an example of a relation which is neither symmetric nor anti-symmetric. Draw its graph.

4) Draw the following relation as a directed graph, and add the *minimum* number of edges necessary to make it into an equivalence relation: $A = \{a, b, c, d, e, f\}$, $R \subseteq A \times A$, $R = \{(a, b), (a, c), (d, e)\}$,**. (This is the transitive, reflexive, and symmetric closure of this relation and is also referred to as "finding the connected components" of the graph). List the equivalence classes of this equivalence relation.**

5) A 'topological sort' of a partial order $P \subseteq A \times A$ is a list of all the elements in A, such that if $(x, y) \in R$, then x appears before y in the list. Write every valid topological sort of the following partial order:
$A = \{pirates, ninjas, robots, zombies\}$, $P : A \to A$,
$P = \{(pirates, ninjas), (zombies, robots), (pirates, robots)\}$
(Feel free to interpret the meaning of the ordering P according to your own beliefs)

6) A *substring* is a piece of a string. For example, "lizatio" would be a substring of "visualization", but "lzt" would not. Like subset, every string is a substring of itself. A *prefix* of a string y is a substring whose first letter is the first letter of y, so "li", "liz" and "liza" are prefixes of "liza", but "iz" is not. If $A = \{x | x$ is a substring of the word "visualization"$\}$, and we define a relation $R = \{(x, y) | x$ is a prefix of $y\}$, is R a partial ordering of A? Is R a total ordering of A? Explain *very carefully* why your answer is correct!

7) Given that $A = \{x \in \mathbb{N} | 1 \le x \le 8\}$ and $R \subseteq A \times A$, $R = \{(x, y) | x = 2^i k$ and $y = 2^i k'$, and k, k' are both odd$\}$

a) Prove that R is an equivalence relation.

b) Write out the set of equivalence classes of R: A/R.

8) For each of the following, First list what properties it has: "reflexive, symmetric, anti-symmetric, transitive". Then label it as a "equivalence relation", "partial (but not total) order", "total order", or "none of the above".

a) $A = \mathbb{N}$, $R \subseteq A \times A$, $R = \{(x, y) | x \le y\}$

b) $A = \mathbb{N}$, $R \subseteq A \times A$, $R = \{(x, y) | x < y\}$)

c) $A =$ the power set of $\{a, b, c, d\}$, $R \subseteq A \times A$, $R = \{(x, y) | x \subseteq y\}$

d) $A = \mathbb{N}$, $R \subseteq A \times A$, $R = \{(x, y) | x = y\}$

e) $A =$People, $R \subseteq A \times A$, $R = \{(x, y)|$x is a relative of y$\}$

f) $A =$People, $R \subseteq A \times A$, $R = \{(x, y)|$x is an ancestor of y, or $x = y\}$

g) $A =$Students, $R \subseteq A \times A$, $R = \{(x, y)|$x is taking the same class as y$\}$

h) $A =$People, $R \subseteq A \times A$, $R = \{(x, y)|$x owes y money$\}$

i) $A =$People, $R \subseteq A \times A$, $R = \{(x, y)|$ x is not older than y$\}$

Proofs:

Remember induction? If we want to prove that some property $P(n)$ is true for *every* $n \in \mathbb{N}$, we first prove that $P(0)$ is true, and then we prove that whenever $P(n)$ is true for some n then $P(n{+}1)$ is also true ($P(n) \Rightarrow P(n{+}1)$). For example, we used this technique to prove that all natural numbers were greater than or equal to 0 ($\forall n \in \mathbb{N}[n \geq 0]$). It turns out that you can use induction on the natural numbers to prove a lot of useful things are true (because a lot of useful things involve the natural numbers), and this type of induction is called "numerical induction". Below are the outlines for a couple proofs using numerical induction. Fill in the blanks to complete the proofs.

9) Easy Proof: Prove that $0 + 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ **for all** $n \in \mathbb{N}$

Base Case (n = 0):

$0 =^? \frac{0(0+1)}{2}$

[***show that this is true. yes, it's very simple.***]

Induction Hypothesis:
Assume that for some n, it's true that $0 + 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Induction step:
Show that if this is the case, it's also true that: $0 + 1 + 2 + 3 + ... + n + (n + 1) = \frac{(n+1)((n+1)+1)}{2}$

$0 + 1 + 2 + 3 + ... + n + (n + 1) =^? \frac{(n+1)((n+1)+1)}{2}$

$(0 + 1 + 2 + 3 + ... + n) + (n + 1) =^? \frac{(n+1)((n+1)+1)}{2}$

$\frac{n(n+1)}{2} + (n + 1) =^? \frac{(n+1)((n+1)+1)}{2}$ (using the Induction Hypothesis on the left side)

[***finish up the algebra to show that both sides of the equation are in fact equal***]

10) Medium Proof: We've said that a good rule of thumb when you're trying to decide whether a relation is transitive is to look at the graph. If the graph is transitive then "whenever it's possible to travel from one point to another by following a series of edges, there should be a single edge connecting them". In other words our rule of thumb is: if $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n) \in R$ **then** $(x_0, x_n) \in R$**. But the original transitive property only says that if:** $(x, y), (y, z) \in R$ **then** $(x, z) \in R$**. Is our rule of thumb ok?**

We can use induction on n **to show that *if the transitive property holds for* R, then our rule of thumb is true *also*, for all** $n \geq 2$**.**

Base Case (n = 2): (the property doesn't really make sense for $n < 2$)

If $(x_0, x_1), (x_1, x_2) \in R$ then $(x_0, x_2) \in R$??

[***show that this is true. remember R is transitive. yes, it's very simple.***]

Induction Hypothesis
Assume that for some n, it's true that: if $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n) \in R$ then $(x_0, x_n) \in R$

Induction step:
Show that if this is the case, then it's also true that: if $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n), (x_n, x_{n+1}) \in R$ then $(x_0, x_{n+1}) \in R$

If $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n), (x_n, x_{n+1}) \in R$ then $(x_0, x_{n+1}) \in R$??

Since all the pairs $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n), (x_n, x_{n+1})$ are in R, it's clearly true that the pairs $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n) \in R$.

[***From here, use the induction hypothesis and the fact that R has the transitive property to show that the pair*** (x_0, x_{n+1}) ***is in fact in*** R]

(***Extra Credit***) **Hard Proof: Use mathematical induction to prove that for all natural numbers** n**, 6 evenly divides** $n^3 - n$**. (In other words, show that for any integer** n**,** $n^3 - n = 6m$**, for *some* integer** m**.**