C241 Homework 7

Due Wednesday, 10/28/09

1) Match the formal definitions for the following properties of relations (and the graphs that represent them) with their informal english equivalents:

i) A relation $R \subseteq A \times A$ is *reflexive* if for every element x in A, there's a pair (x, x) in R. When you draw R as a directed graph, this means that each point in the graph has an arrow pointing to itself.

c) $\forall x \in A[x \in A \Rightarrow (x, x) \in R]$

ii) A relation $R \subseteq A \times A$ is *symmetric* if, for every pair (x, y) that appears in R (where x $\neq$ y), the reverse of that pair, (y, x), also appears in R. When you draw R as a directed graph, this means that if there's an arrow from x to y, then there is also an arrow from y to x.

a) $\forall (x, y) \in R$ such that $x \neq y[(x, y) \in R \Rightarrow (y, x) \in R]$

iii) A relation $R \subseteq A \times A$ is *anti-symmetric* if, for every pair (x, y) that appears in R (where x $\neq$ y), the reverse of that pair, (y, x), does **not** appear in R. When you draw R as a directed graph, this means that if there's an arrow from x to y, then there is **not** an arrow from y to x.

h) $\forall (x, y) \in R$ such that $x \neq y[(x, y) \in R \Rightarrow (y, x) \notin R]$

iv) A relation $R \subseteq A \times A$ is *transitive* if, whenever you have two pairs such that the last element of one pair is the first element of the other: (x, y) (y, z), you also have a pair: (x, z). When you draw R as a directed graph, this means that whenever you can get from one point to another by following a sequence of arrows, there is also a single arrow that connects the first point to the second.

g) $\forall (x, y), (y, z) \in R[(x, y), (y, z) \in R \Rightarrow (x, z) \in R]$

v) A relation $R \subseteq A \times A$ is an *equivalence relation* if it is reflexive, symmetric and transitive. When you draw R as a directed graph, this means that the graph will partition the points in A into disjoint clumps of fully connected points (these clumps are referred to as equivalence classes).

d) R is transitive, symmetric, and reflexive.

vi) The *equivalence class* of an element $x \in A$ under an equivalence relation R, is the set of all elements in A which R pairs with x. This is written as $[x]_R$. Since equivalence relations are reflexive, the equivalence class of x will always include x. When you draw R as a directed graph, all the elements which are clumped with x are in the same equivalence class as x.

b) Given that R is an equivalence relation: $\{y \in A | (x, y) \in R\}$

vii)The set of equivalences classes that an equivalence relation R breaks A into is written as A/R. This will be a partition of A, which means that it will be a set of subsets of A, and each element of A will appear in exactly one of the subsets.

e) A partition of S into distinct sets: $S_1, S_2, ...S_n$ such that if $x \in S_i$ and $(x, y) \in R$ then $y \in S_i$.

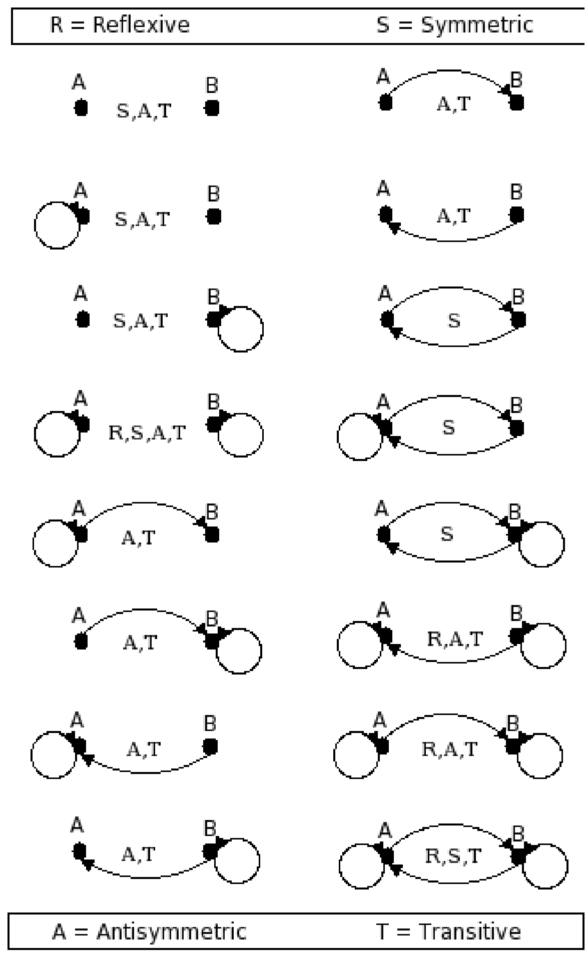
viii) A relation $R \subseteq A \times A$ is a *partial ordering* if it is anti-symmetric and transitive. $\subseteq$ is a partial ordering over any set A.

f) R is transitive and anti-symmetric.

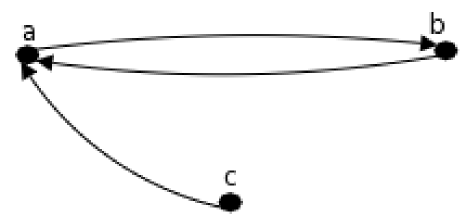
ix) A relation $R \subseteq A \times A$ is a *total ordering* if it is reflexive, anti-symmetric and transitive, and if for any two elements x, y in A, x and y are related to each other by R: either $(x, y) \in R$ or $(y, x) \in R$. "$\geq$" is a total ordering over the natural numbers.

i) R is transitive and anti-symmetric,
and $\forall x, y \in A[(x, y) \in R$ or $(y, x) \in R]$

2) Given A = {a, b}, draw the graphs for *every* possible relation R such that $R \subseteq A \times A$. Label each graph with the properties it has: reflexive, symmetric, anti-symmetric, and/or transitive. Make sure you pay careful attention to the definitions above, this can be tricky.

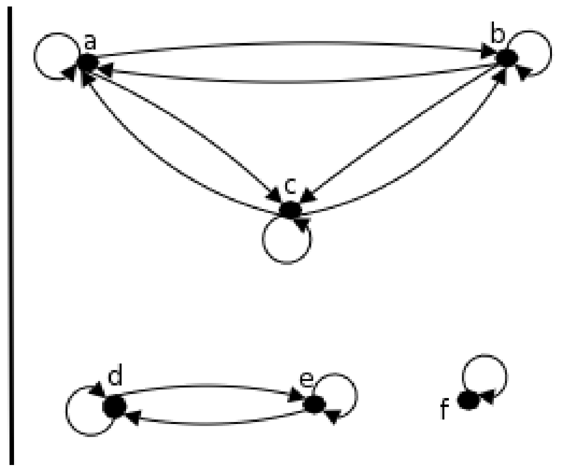


3) Give an example of a relation which is neither symmetric nor anti-symmetric. Draw its graph.



4) Draw the following relation as a directed graph, and add the *minimum* number of edges necessary to make it into an equivalence relation: $A = \{a, b, c, d, e, f\}$, $R \subseteq A \times A$, $R = \{(a, b), (a, c), (d, e)\}$,**. (This is the transitive, reflexive, and symmetric closure of this relation and is also referred to as "finding the connected components" of the graph). List the equivalence classes of this equivalence relation**.

The equivalence classes in this graph are: {a, b, c}, {d, e}, {f}.



5) A 'topological sort' of a partial order $P \subseteq A \times A$ is a list of all the elements in A, such that if $(x, y) \in R$, then x appears before y in the list. Write every valid topological sort of the following partial order:
$A = \{pirates, ninjas, robots, zombies\}$, $P : A \rightarrow A$,
$P = \{(pirates, ninjas), (zombies, robots), (pirates, robots)\}$
(Feel free to interpret the meaning of the ordering P according to your own beliefs)

P = pirates, N = ninjas, Z = zombies, R = Robots:

PZNR

PZRN

ZPNR

ZPRN

PNZR

6) A *substring* is a piece of a string. For example, "lizatio" would be a substring of "visualization", but "lzt" would not. Like subset, every string is a substring of itself. A *prefix* of a string y is a substring whose first letter is the first letter of y, so "li", "liz" and "liza" are prefixes of "liza", but "iz" is not. If $A = \{x | x$ is a substring of the word "visualization"$\}$, and we define a relation $R = \{(x, y) | x$ is a prefix of $y\}$, is R a partial ordering of A? Is R a total ordering of A? Explain *very carefully* why your answer is correct!

To prove that R is a partial order, we need to show that it's antisymmetric and transitive.

Antisymmetric: if $a \neq b$ and (a, b) is a pair in R, then (b, a) cannot be in R. So, if (x, y) is a pair in R, that means that x is a prefix of y, (where x and y are both substrings of the word "visualization"), and that means that x is either shorter than y or x is the same string as y. We're only concerned about edges between two different points when we're checking asymmetry. So we'll just look at the cases where x is shorter than y. If y is longer than x, y can't be a prefix of x. And thus, if $x \neq y$ and (x, y) is a pair in R, then (y, x) cannot be in R. R is anti-symmetric.

Transitive: if $(a, b), (b, c)$ are two pairs in R, then (a, c) must also be a pair in R. If x is a prefix of y, then x and y start with the same letter, and x is a substring of y. If y is a prefix of z, then y and z start with the same letter (which is also the same first letter as x), and y is a substring of z (which means that x is a substring of z

too, since x is a substring of y). So, x is a prefix of z. In other words, if $(x, y)(y, z)$ are two pairs in R, then $(x, z0$ must also be a pair in R. R is transitive.

If R were a total order, then for any two elements x, y of A (any two substrings of the word "visualization"), either the pair (x, y) would be in R (x would be a prefix of y), or the pair (y, x) would be in R (y would be a prefix of x). Sometimes this is phrased as saying "Any pair of strings in A is comparable under R." However, "vis" and "tion" are two substrings of visualization, so they're both elements of A. But neither is a prefix of the other. So $(vis, tion)$ isn't in R, and neither is $(tion, vis)$. Thus not all pairs of elements in A are comparable under R, and R is not a total ordering of A.

7) Given that $A = \{x \in \mathbb{N} | 1 \leq x \leq 8\}$ and $R \subseteq A \times A$, $R = \{(x, y) | x = 2^i k$ and $y = 2^i k'$, and k, k' are both odd$\}$

a) Prove that R is an equivalence relation.

We need to show that A has three properties: reflexive, symmetric and transitive.

Reflexive: For any $n \in A$, We'll have $n = 2^i k$ (for some value of i and k), and we also get $n = 2^i k'$ (for the same i as before and letting $k' = k$). So we have $(n, n) \in R$.

Symmetric: For any $x, y \in A$ such that $(x, y) \in R$, we know that $x = 2^i k$ and $y = 2^i k'$. So if we swap the values of k and k', we get $y = 2^i k$, $x = 2^i k'$, and so $(y, x) \in R$

Transitive: If we have $(x, y) \in R$ and $(y, z) \in R$, we know that there's some value of i such that x, y and z are each equal to 2^i times some odd number. Say $x = 2^i k_x$ (and similar for y and z). Then set $k = k_x$, $k' = k_z$, and we get $(x, z) \in R$.

b) Write out the set of equivalence classes of R: A/R.

So, first let's see what the equivalence class for 1 is (ie, $[1]_R$).
$1 = 2^0 \times 1$, so 1 will be related to anything of the form $2^0 \times k'$ where k' is an odd number between 1 and 8.
Thus $[1]_R = \{1, 3, 5, 7\}$ (for $k' = 1, 3, 5$ and 7). That takes care of those numbers
(if $y \in [x]_R$ then $[y]_R = [x]_R$, so by finding the equivalence class for 1, we've found the equivalence classes for $3, 5$ and 7 as well. You should understand why this is the case. Look up at the graph in problem 7 for inspiration if

you're confused).
Anyway, $2 \notin [1]_R$, so let's see what $2'$s equivalence class looks like.
$2 = 2^1 \times 1$. So 2 will be related to anything of the form $2^1 \times k'$ where k' is an odd number between 1 and 8.
Of course R is reflexive: since k' can be 1, the same as k, we know that $2 \in [2]_R$
Also, since $6 = 2^1 \times 3$, we know that $6 \in [2]_R$.
Is there anything else? The next possibility for k' is 5, but $2^1 \times 5 = 10$, which is outside our range of 1 to 8.
So the only things related to 2 are 2 and 6: $[2]_R = \{2, 6\}$.
What's left? We haven't dealt with 4 or 8 yet, and since this is an equivalence relation, we know that each element in the domain is at least related to itself (reflexive).
$4 = 2^2 \times 1$, so since it's possible for $k' = 1$, we know 4 is related to itself.
However $2^2 \times 3 = 12$, which is outside our range, so the *only* thing related to 4 is 4: $[4]_R = \{4\}$.
Something similar is true for $8 = 2^3 \times 1$, $[8]_R = \{8\}$.
Thus, in total, we have four equivalence classes: $\{1, 3, 5, 7\}, \{2, 6\}, \{4\}, \{8\}$.
We write the set of equivalence classes that R breaks A into as $A/R = \{\{1, 3, 5, 7\}, \{2, 6\}, \{4\}, \{8\}\}$

8) For each of the following, First list what properties it has: "reflexive, symmetric, anti-symmetric, transitive". Then label it as a "equivalence relation", "partial (but not total) order", "total order", or "none of the above".

a) $A = \mathbb{N}$, $R \subseteq A \times A$, $R = \{(x, y) | x \leq y\}$ Reflexive, Anti-symmetric, Transitive: Total Order (because if we pick *any* x and any y from A, we know that either $x \leq y$ (so $(x, y) \in R$), or $y \leq x$ (so $(y, x) \in R$)).)

b) $A = \mathbb{N}$, $R \subseteq A \times A$, $R = \{(x, y) | x < y\}$) Anti-symmetric, Transitive: Nothing

c) $A =$ the power set of $\{a, b, c, d\}$, $R \subseteq A \times A$, $R = \{(x, y) | x \subseteq y\}$ Reflexive (a set is always a subset of itself), Anti-symmetric, Transitive: Partial (but not total) Order (it's not total because, for example, if $x = \{a, b\}$ and $y = \{c, d\}$ then neither (x, y) nor (y, x) are in R)

d) $A = \mathbb{N}$, $R \subseteq A \times A$, $R = \{(x, y)|x = y\}$ Reflexive, Symmetric, Transitive... and, weirdly, technically also Anti-symmetric (because there will never be an edge between points x and y where $x \neq y$). If we were to draw this out as a graph, we'd have one point for every natural number, and each point would have a loop pointing to itself. So, technically, this is an Equivalence Relation, and a (basically useless) Partial Order. But not a total order, for obvious reasons.

e) $A =$People, $R \subseteq A \times A$, $R = \{(x, y)|\text{x is a relative of y}\}$ Reflexive (debatably), Symmetric, Transitive: Equivalence Relation (debatably, due to reflexivity). The equivalence classes here would be families

f) $A =$People, $R \subseteq A \times A$, $R = \{(x, y)|\text{x is an ancestor of y, or } x = y\}$ Reflexive, Symmetric, Transitive: Partial Order (but not total because it's easy to find an $x, y \in A$ such that neither (x, y) nor (y, x) is in R (for example any two people who aren't related).

g) $A =$Students, $R \subseteq A \times A$, $R = \{(x, y)|\text{x is taking the same class as y}\}$ Reflexive, Symmetric: Nothing. Not Transitive, because x and y could both be taking C241, and y and z could both be taking 343, but x and z might not have any classes in common. If everyone only took one course a semester, this would be transitive, an equivalence relation, and the equivalence classes would be... classes.

h) $A =$People, $R \subseteq A \times A$, $R = \{(x, y)|\text{x owes y money}\}$ Anti-symmetric (debatably): Nothing. Not reflexive, because you don't usually owe yourself money (depending on how you think about managing your savings), and not transitive because debts are usually not considered transitive (if john owes you five bucks, and you owe capitol one 300 dollars, john does not owe capital one five bucks).

i) $A =$People, $R \subseteq A \times A$, $R = \{(x, y)|\text{ x is not older than y}\}$ Reflexive (you're not older than yourself), Transitive. Not strictly anti-symmetric, because if x and y are twins (two different people with the same age), then we'll have (x, y) and (y, x) in R.

Proofs:

Remember induction? If we want to prove that some property $P(n)$ is true for *every* $n \in \mathbb{N}$, we first prove that $P(0)$ is true, and then we prove that whenever $P(n)$ is true for some n then $P(n+1)$ is also true ($P(n) \Rightarrow P(n+1)$). For example, we used this technique to prove that all natural numbers were greater than or equal to 0 ($\forall n \in \mathbb{N}[n \geq 0]$). It turns out that you can use induction on the natural numbers to prove a lot of useful things are true (because a lot of useful things involve the natural numbers), and this type of induction is called "numerical induction". Below are the outlines for a couple proofs using numerical induction. Fill in the blanks to complete the proofs.

9) Easy Proof: Prove that $0 + 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$

Base Case (n = 0):

$0 =^? \frac{0(0+1)}{2}$

$0 =^? \frac{0}{2}$

$0 = 0$

Induction Hypothesis:
Assume that for some n, it's true that $0 + 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Induction step:
Show that if this is the case, it's also true that: $0 + 1 + 2 + 3 + ... + n + (n + 1) = \frac{(n+1)((n+1)+1)}{2}$

$0 + 1 + 2 + 3 + ... + n + (n + 1) =^? \frac{(n+1)((n+1)+1)}{2}$

$(0 + 1 + 2 + 3 + ... + n) + (n + 1) =^? \frac{(n+1)((n+1)+1)}{2}$

$\frac{n(n+1)}{2} + (n + 1) =^? \frac{(n+1)((n+1)+1)}{2}$ (using the Induction Hypothesis on the left side)

$$\frac{n(n+1)}{2} + \frac{2(n+1)}{2} =^? \frac{(n+1)((n+1)+1)}{2}$$ (get a common denominator)

$$\frac{n(n+1) + 2(n+1)}{2} =^? \frac{(n+1)(n+2)}{2}$$ (basic algebra)

$$\frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+2)}{2} \text{ (factor out (n + 1) on left hand side)}$$

10) Medium Proof: We've said that a good rule of thumb when you're trying to decide whether a relation is transitive is to look at the graph. If the graph is transitive then "whenever it's possible to travel from one point to another by following a series of edges, there should be a single edge connecting them". In other words our rule of thumb is: if $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n) \in R$ then $(x_0, x_n) \in R$. But the original transitive property only says that if: $(x, y), (y, z) \in R$ then $(x, z) \in R$. Is our rule of thumb ok?

We can use induction on n to show that *if the transitive property holds for R*, then our rule of thumb is true *also*, for all $n \geq 2$.

Base Case (n = 2): (the property doesn't really make sense for $n < 2$)

If $(x_0, x_1), (x_1, x_2) \in R$ then $(x_0, x_2) \in R$??

We've said that R is transitive, and the transitive property states that if $(x_0, x_1), (x_1, x_2) \in R$, we must also have $(x_0, x_2) \in R$.

Induction Hypothesis
Assume that for some n, it's true that: if $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n) \in R$ then $(x_0, x_n) \in R$

Induction step:
Show that if this is the case, then it's also true that: if $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n), (x_n, x_{n+1}) \in R$ then $(x_0, x_{n+1}) \in R$

If $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n), (x_n, x_{n+1}) \in R$ then $(x_0, x_{n+1}) \in R$??

Since all the pairs $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n), (x_n, x_{n+1})$ are in R, it's clearly true that the pairs $(x_0, x_1), (x_1, x_2), (x_2, x_3)...(x_{n-1}, x_n) \in R$.

From the induction hypothesis, then, we know that $(x_0, x_n) \in R$.

We've also still got that $(x_n, x_{n+1}) \in R$. So, by the regular transitive property on these two edges, we must have $(x_0, x_{n+1}) \in R$. And that completes the proof.

(*Extra Credit*) Hard Proof: Use mathematical induction to prove that for all natural numbers n, 6 evenly divides $n^3 - n$. (In other words, show that for any integer n, $n^3 - n = 6m$, for *some* integer m.

Base Case: $n = 0$

$0^3 - 0 = 0 = 6 \times 0$

Induction Hypothesis: Assume that $n^3 - n = 6m$, for some integer m.

Induction Step: Show that this implies that $(n+1)^3 - (n+1) = 6m'$, for some integer m' (note $m \neq m'$, so we can't just use m for both).

$(n+1)^3 - (n+1) =^? 6m'$

$n^3 + 3n^2 + 3n + 1 - (n+1) =^? 6m'$

$n^3 + 3n^2 + 3n + 1 - n - 1 =^? 6m'$

$(n^3 - n) + 3n^2 + 3n =^? 6m'$

$(n^3 - n) + 3(n^2 + n) =^? 6m'$

$(n^3 - n) + 3(n(n+1)) =^? 6m'$

Since n and $n+1$ are successive integers, *one* of them is bound to be even. And the product of any number and an even number is even. So we know that for some integer k, $n(n+1) = 2k$.

$(n^3 - n) + 3(2k) =^? 6m'$

$6m + 6k =^? 6m'$ Induction Hypothesis

$6(m+k) =^? 6m'$ Since m and k are both integers, this means that $m' = m + k$ is an integer, and thus $(n+1)^3 - (n+1)$ divided by 6 is the integer $m + k$.... $(n+1)^3 - (n+1)$ is evenly divisible by 6.