C241 Homework 8

Due Wednesday, 11/4/09

1) A = {1, 2, 3, 4, 5, 6}
and $R \subseteq A \times A$, $R = \{(x, y)|x - y$ is evenly divisible by 3$\}$ (we consider 0 to be evenly divisible by 3). Show R is an equivalence relation by showing that it is reflexive, symmetric, and transitive. Then write out the set A/R.

2) Remember that a relation $R \subseteq A \times A$ is a partial order if it is: reflexive, anti-symmetric and transitive. And, a partial order is called a "total order" if it is reflexive, anti-symmetric and transitive *and* for every two elements $x, y \in A$, either $(x, y) \in R$ or $(y, x) \in R$.

a)Pick a set A and create a relation $R \subseteq A \times A$ such that R is a partial order, but not a total order. Prove that your relation is reflexive, symmetric, transitive, and that there's at least two elements $x, y \in A$ such that neither (x, y) nor (y, x) is in R.

b)Pick a set B and create a relation $Q \subseteq B \times B$ such that Q is a total order. Prove that your relation is reflexive, symmetric, transitive, and show that when you pick *any* two elements $x, y \in A$ either (x, y) or (y, x) will be in Q.

3)The following are mistakes that often show up in numerical induction proofs. Clearly describe what is wrong with each one.

a) $\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 + (n + 1)$

b) $(n + 2)! = n! + 2!$

c) $\sum_{i=0}^{n} i^2 = \sum_{i=0}^{n} i^2 + (n + 1)^2$

d)$(2n)^2 = 2n^2$

e)$\frac{(n+1)2n}{6} + n^3(n + 1)^2 = \frac{(n+1)(n+2)}{6}$ can be simplified to: $\frac{2n}{6} + n^3 = \frac{(n+2)}{6}$

f) **Prove:** All numbers are prime.

Base Case: n = 1. 1 is a prime number, so the base case holds.

Induction Step: Let's assume n is a prime number (this is our induction hypothesis), then we need to show that this implies n+1 is also prime. Since n = 1, $1 + 1 = 2$, and 2 is prime, the induction step holds. Thus all numbers are prime.
Note: Please explain how the technique used in this proof is incorrect; don't just offer a counter-example

4) Using only the Distributive law $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$, the Associative law, and Numerical Induction, prove that the distributive property is valid for n variables $\forall n \in \mathbb{N}$: $p \wedge (q_1 \vee q_2 \vee q_3 \vee ... \vee q_n) \equiv (p \wedge q_1) \vee (p \wedge q_2) \vee (p \wedge q_3) \vee ... \vee (p \wedge q_n)$

5) Use numerical induction to prove that $\forall n \in \mathbb{N}$: $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

6) Use numerical induction to prove that $\forall n \in \mathbb{N}$: $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

7) Prove the following two inequalities.

a) Use numerical induction to prove: $2^n > 2n + 1$ for $n \geq 3$. (Your base case here will be $n = 3$. You can use any basic properties of inequalities).

b) Use numerical induction to prove: $2^n > n^2$ for $n \geq 5$. (Your base case here will be $n = 5$, and the result in (a) may be useful. You can use any basic properties of inequalities).

8) Use numerical induction to prove that:
for all $n \in \mathbb{N}$, the function $f(n) = 3^n$.

$$f : \mathbb{N} \to \mathbb{N}$$

$$f(0) = 1$$

$$\forall n > 0, \quad f(n) = 3 \times f(n-1)$$

9) Use numerical induction to prove that:
for all $n \in \mathbb{N}$**:** $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$

10) (Bonus) Explain carefully and clearly why math is not *actually* broken.