C241 Homework 8

Due Wednesday, 11/4/09

1) A = {1, 2, 3, 4, 5, 6}
and $R \subseteq A \times A$, $R = \{(x, y)|x - y$ is evenly divisible by 3$\}$ (we consider 0 to be evenly divisible by 3). Show R is an equivalence relation by showing that it is reflexive, symmetric, and transitive. Then write out the set A/R.

Reflexive: for any $x \in A$, $x - x = 0$, and since 0 is divisible by 3, this means that for any $x \in A$, $(x, x) \in R$. So R is reflexive.

Symmetric: If $(x, y) \in R$ and $x \neq y$, then that means that $x - y = 3m$ for some integer m (in other words $x - y$ must be equal to 3 times something, and we're calling the something 'm'). And, if $x - y = 3m$, then $y - x = -3m = 3 \times -m$. So $y - x$ is also equal to 3 times an integer, and this mean $(y, x) \in R$. So R is symmetric.

Transitive: Using a similar reasoning as above, if we have $(x, y) \in R$ and $(y, z) \in R$, then we know $(x - y) = 3m$ and $(y - z) = 3n$, for some integers m and n. In this case $x - z = (x - y) + (y - z) = 3m + 3n = 3(m + n)$, which means $x - z$ is equal to 3 times an integer. So we have $(x - z) \in R$ and R is transitive.

Since R is reflexive, symmetric and transitive, R must be an equivalence relation. We use R to partition A into subgroups of items that are all related to each other by R (called equivalence classes). We write this as $R/A = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$

2) Remember that a relation $R \subseteq A \times A$ is a partial order if it is: reflexive, anti-symmetric and transitive. And, a partial order is called a "total order" if it is reflexive, anti-symmetric and transitive *and* for every two elements $x, y \in A$, either $(x, y) \in R$ or $(y, x) \in R$.

> **a) Pick a set A and create a relation $R \subseteq A \times A$ such that R is a partial order, but not a total order. Prove that your relation is reflexive, anti-symmetric, transitive, and that there's at least two elements $x, y \in A$ such that neither (x, y) nor (y, x) is in R.**
>
> $A = \{Sugar\ and\ Spice, Mcdonalds, Wendys, Burger\ King, Marsh, Krogers, Warlmart, CollegeM$
>
> $R = \{(x, y)$ | either $x = y$, or *all* instances of x in Bloomington are larger than all instances of y in Bloomington $\}$
>
> Reflexive: for all $z \in A$, we'll have $(z, z) \in R$, because obviously $z = z$ and the relation states that if x equals y, then $(x, y) \in R$.

Anti-symmetric: If $x \neq y$ and $(x, y) \in R$, then we know that all instances of x are larger than all instances of y (in Bloomington), and thus it *can't* be the case that all instances of y are bigger than all instances of x, and so $(y, x) \notin R$. Thus R is anti-symmetric.

Transitive: If $(x, y) \in R$ and $(y, z) \in R$, then we know that every instance of x is bigger than every instance of y (or $x = y$), which are themselves bigger than every instance of z (or $y = z$). So every instance of x is bigger than every instance of z (or $x = z$), and we have $(x, z) \in R$. Thus R is transitive.

Not Total: Since R is reflexive, anti-symmetric, transitive it is a Partial Order. And it's obvious that we'll have edges like $(CollegeMall, McDonalds)$ and $(Walmart, Marsh)$ in R. But when it comes to $x = McDonalds$ and $y = BurgerKing$ it's more interesting. Then all Burger Kings aren't larger than all McDonalds, and all McDonalds aren't larger than all Burger Kings, so we get $(x, y) \notin R$ and $(y, x) \notin R$, and the partial order must not be total.

b)Pick a set B and create a relation $Q \subseteq B \times B$ such that Q is a total order. Prove that your relation is reflexive, anti-symmetric, transitive, and show that when you pick *any* two elements $x, y \in A$ either (x, y) or (y, x) will be in Q.

$B =$ the set of C241 students

$Q = \{(x, y) \mid$ if you put x and y in alphabetical order by their full names, then $x = y$ or x comes before $y.\}$

Reflexive: Since $x = x$, we'll have $(x, x) \in Q$ for all $x \in B$, so Q is Reflexive.

Anti-symmetric: If we have $x \neq y$ and $(x, y) \in Q$, then we know that x's name comes before y's in alphabetical order, and thus y's name can't come before x, so $(y, x) \notin Q$. So Q is anti-symmetric.

Transitive: If we have $(x, y) \in Q$ and $(y, z) \in Q$, then x's name comes before y (or $x = y$), and y's name comes before z's (or $y = z$), and thus x's name comes before z's (or $x = z$). Thus Q is transitive.

Total: Since Q is reflexive, anti-symmetric and transitive, it'll be a partial order. Now I'll show that it's a total order. We've got a couple Andrews and a couple Mohammeds, but no two different people in the class have exactly the same full name. So, if I pick any two students x and y from the class, either x's name is going to come before y's in alphabetical order (so $(x, y) \in Q$), or it'll be the other way around (so $(y, x) \in Q$). This means that Q is a total order. A good way to think about total orders is that if we had every student in the class line up in alphabetical order by their full name, we'd end up with everyone standing in a single long straight line and there'd be no question what each student's place in that line should be (in other words, if Q is a total order, there's only one valid topological sort for B using Q). By contrast, if we tried to list out the stores from part (a) in order by their size, we wouldn't know whether McDonalds should come before or after Burger King (or Wendy's for that matter). (Since R is not a total order, there's several valid topological sorts of Q using R).

3)The following are mistakes that often show up in numerical induction proofs. Clearly describe what is wrong with each one.

a) $\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 + (n+1)$

This would be correct if the $(n+1)$ on the right was squared.
$\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 + (n+1)^2$ since:

$\sum_{i=0}^{n+1} i^2 =$
$1^2 + 2^2 + 3^2 + ... + n^2 + (n+1)^2 =$
$(1^2 + 2^2 + 3^2 + ... + n^2) + (n+1)^2 =$
$\sum_{i=0}^{n} i^2 + (n+1)^2$

b) $(n+2)! = n! + 2!$

This is incorrect because factorial does not distribute.
Note: $(n+2)! = (n+2)(n+1)n! \neq n! + 2!$

c) $\sum_{i=0}^{n} i^2 = \sum_{i=0}^{n} i^2 + (n+1)^2$

This is clearly incorrect, since the right hand side is a postive amount, $(n+1)^2$, greater than the left hand side.
As we said in 1a), a correct equality would be: $\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 + (n+1)^2$

d)$(2n)^2 = 2n^2$

Unlike factorials, exponents *do* distribute: $(2n)^2 = 2^2n^2 = 4n^2$

e) $\frac{(n+1)2n}{6} + n^3(n+1)^2 = \frac{(n+1)(n+2)}{6}$ can be simplified to: $\frac{2n}{6} + n^3 = \frac{(n+2)}{6}$

In this case, $(n+1)$ should have been divided out from every term on both sides of the equality. Instead, $(n+1)^2$ was divided out from the $n^3(n+1)^2$ term on the left hand side, while only $(n+1)$ was divided out everywhere else. If it had been done properly, the simplified equality would be:

$\frac{2n}{6} + n^3(n+1) = \frac{(n+2)}{6}$

f) **Prove:** All numbers are prime.

Base Case: n = 1. 1 is a prime number, so the base case holds.

Induction Step: Let's assume n is a prime number (this is our induction hypothesis), then we need to show that this implies n+1 is also prime. Since n = 1, $1 + 1 = 2$, and 2 is prime, the induction step holds. Thus all numbers are prime.

Despite the nice format, this is actually not an inductive proof. An inductive proof uses two parts, a base case, and a "step rule", to prove that a claim is true in all cases. For instance, if I knew that whenever *any* day was rainy, the next day would also be rainy (the step rule), and I also knew that the very first day was rainy (the base case), then I've shown that all days are rainy, without having to check each day individually (Since if the first day is rainy, my step rule tells me the second day will be rainy too. And if the second day is rainy, my step rule tells me the third day will also be rainy, and so on...). When you're doing an inductive proof for a claim you prove two things: that the claim is true for the base case, and that if it is true for *any* number n, then it is true for the next number (n + 1)... and those two proofs show that the claim is true for all numbers. So for the base case you choose a specific starting value, say n = 1, and then show that the claim is true if n has the value 1. This was done correctly in the proof above. For the step rule, however, you have to show that *for any number n*, if the claim is true for n, then it's true (n + 1). For the proof above, this would mean proving that if n is any prime number, then (n + 1) is a prime number too, which is clearly not true. The proof above avoids this by just plugging in a specific value for n, $n = 1$; which means that this proof does not have a valid step rule, and thus rather than proving the claim true for all n, it just proves the claim for $n = 1$ and $n = 2$.

4) Using only the Distributive law $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$, the Associative law, and Numerical Induction, prove that the distributive property is valid for n variables $\forall n \in \mathbb{N}$: $p \wedge (q_1 \vee q_2 \vee q_3 \vee ... \vee q_n) \equiv (p \wedge q_1) \vee (p \wedge q_2) \vee (p \wedge q_3) \vee ... \vee (p \wedge q_n)$

Base Case (n = 2): $p \wedge (q_1 \vee q_2) \Leftrightarrow (p \wedge q_1) \vee (p \wedge q_2)$

This is just the normal two variable Distributive Law.

Induction Hypothesis: Assume that the Distributive Law works for n variables, so assume: $p \wedge (q_1 \vee q_2 \vee q_3 \vee ... \vee q_n) = (p \wedge q_1) \vee (p \wedge q_2) \vee (p \wedge q_3) \vee ... \vee (p \wedge q_n)$

Induction Step (n + 1): If the Distributive Law works for n variables, does it work for $n + 1$?

$p \wedge (q_1 \vee q_2 ... \vee q_n \vee q_{n+1})$

$\Leftrightarrow p \wedge ((q_1 \vee q_2 ... \vee q_n) \vee q_{n+1})$ Associative Law

$\Leftrightarrow (p \wedge (q_1 \vee q_2 ... \vee q_n)) \vee (p \wedge q_{n+1})$ Distributive Law (the two variable version)

$\Leftrightarrow ((p \wedge q_1) \vee (p \wedge q_2) ... \vee (p \wedge q_n)) \vee (p \wedge q_{n+1})$ This was our Induction Hypothesis.

$\Leftrightarrow (p \wedge q_1) \vee (p \wedge q_2) ... \vee (p \wedge q_n) \vee (p \wedge q_{n+1})$ Associative Law

5) Use numerical induction to prove that $\forall n \in \mathbb{N}$: $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Base Case (n = 0):

$$\sum_{i=0}^{0} i =^{?} \frac{0(0+1)}{2}$$

$0 = 0$

Induction Hypothesis: Assume $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Induction Step: Then show that this implies: $\sum_{i=0}^{(n+1)} i = \frac{(n+1)((n+1)+1)}{2}$

$$\sum_{i=0}^{(n+1)} i =^? \frac{(n+1)((n+1)+1)}{2}$$

$$\sum_{i=0}^{n} i + (n+1) =^? \frac{(n+1)((n+1)+1)}{2} \text{ (expand the summation)}$$

$$\frac{n(n+1)}{2} + (n+1) =^? \frac{(n+1)((n+1)+1)}{2} \text{ (use the Induction Hypothesis)}$$

$$\frac{n(n+1)}{2} + \frac{2(n+1)}{2} =^? \frac{(n+1)((n+1)+1)}{2} \text{ (get a common denominator)}$$

$$\frac{n(n+1)+2(n+1)}{2} =^? \frac{(n+1)(n+2)}{2} \text{ (basic algebra)}$$

$$\frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+2)}{2} \text{ (factor out (n + 1) on left hand side)}$$

6) Use numerical induction to prove that $\forall n \in \mathbb{N}$: $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Base Case (n = 0):

$$\sum_{i=0}^{0} i^2 =^? \frac{0(0+1)(2(0)+1)}{6}$$

$0 = 0$

Induction Hypothesis: Assume $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Induction Step: Then show that this implies: $\sum_{i=0}^{(n+1)} i^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$

$$\sum_{i=0}^{(n+1)} i^2 =^? \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

$$\sum_{i=0}^{(n+1)} i^2 =^? \frac{(n+1)(n+2)(2n+3)}{6} \text{ (basic algebra)}$$

$$\sum_{i=0}^{n} i^2 + (n+1)^2 =^? \frac{(n+1)(n+2)(2n+3)}{6} \text{ (expand the summation)}$$

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 =^? \frac{(n+1)(n+2)(2n+3)}{6} \text{ (Induction Hypothesis)}$$

$$\frac{n(2n+1)}{6} + (n+1) =^? \frac{(n+2)(2n+3)}{6} \text{ (cancel out a (n + 1))}$$

$$\frac{n(2n+1)}{6} + \frac{6(n+1)}{6} =^? \frac{(n+2)(2n+3)}{6} \text{ (get a common denominator)}$$

$$\frac{n(2n+1)+6(n+1)}{6} =^? \frac{(n+2)(2n+3)}{6} \text{ (combine fractions)}$$

$$\frac{2n^2+n+6n+6)}{6} =^? \frac{2n^2+3n+4n+6}{6} \text{ (multiply everything out)}$$

$$\frac{2n^2+7n+6)}{6} = \frac{2n^2+7n+6}{6} \text{ (simplify)}$$

7) Prove the following two inequalities.

a) Use numerical induction to prove: $2^n > 2n+1$ for $n \geq 3$. (Your base case here will be $n = 3$. You can use any basic properties of inequalities).

Base Case: $n = 3$
$2^3 >^? 2 * 3 + 1$
$8 > 7$

Induction Hypothesis: Assume $2^n > 2n + 1$ for some $n \geq 3$
Induction Step: Show that this implies: $2^(n + 1) > 2(n + 1) + 1$
$2^(n + 1) >^? 2(n + 1) + 1$
$2 \times 2^n >^? 2(n + 1) + 1$
$2^n + 2^n >^? 2n + 2 + 1$
$2^n + 2^n >^? 2n + 1 + 2$
We know that $2^n > 2n + 1$, by our induction hypothesis. And, it's safe to assume that for $n \geq 3$, $2^n > 2$. So, since both terms on the left are greater than the terms on the right, the left hand side is greater.

b) Use numerical induction to prove: $2^n > n^2$ for $n \geq 5$. (Your base case here will be $n = 5$, and the result in (a) may be useful. You can use any basic properties of inequalities).

Base Case: $n = 5$

$2^5 >^? 5^2$

$32 > 25$

Induction Hypothesis: Assume $2^n > n^2$ for some $n \geq 5$

Induction Step: Then show that this implies: $2^(n + 1) > (n + 1)^2$

$2^(n + 1) >^? (n + 1)^2$

$2 \times 2^n >^? n^2 + 2n + 1$

$2^n + 2^n >^? n^2 + 2n + 1$

We know that $2^n > n^2$, by our induction hypothesis. And, very conveniently, we just got done proving that $2^n > 2n + 1$. So, since both terms on the left are greater than the terms on the right, the left hand side is greater.

8) Use numerical induction to prove that:
for all $n \in \mathbb{N}$, the function $f(n) = 3^n$.

$f : \mathbb{N} \to \mathbb{N}$	
	$f(0) = 1$
$\forall n > 0,$	$f(n) = 3 \times f(n - 1)$

Base Case: $n = 0$

$f(0) =^? 3^0$

$1 = 1$

Induction Hypothesis: Assume $f(n) = 3^n$ for some $n \in mathbbmN$

Induction Step: Then show that this implies: $f(n + 1) = 3^{n+1}$

$f(n + 1) =^? 3^{n+1}$

$3 \times f(n + 1 - 1) =^? 3^{n+1}$ (using the function definition)

$3 \times f(n) =^? 3^{n+1}$

$3 \times 3^n =^? 3^{n+1}$ (induction hypothesis)

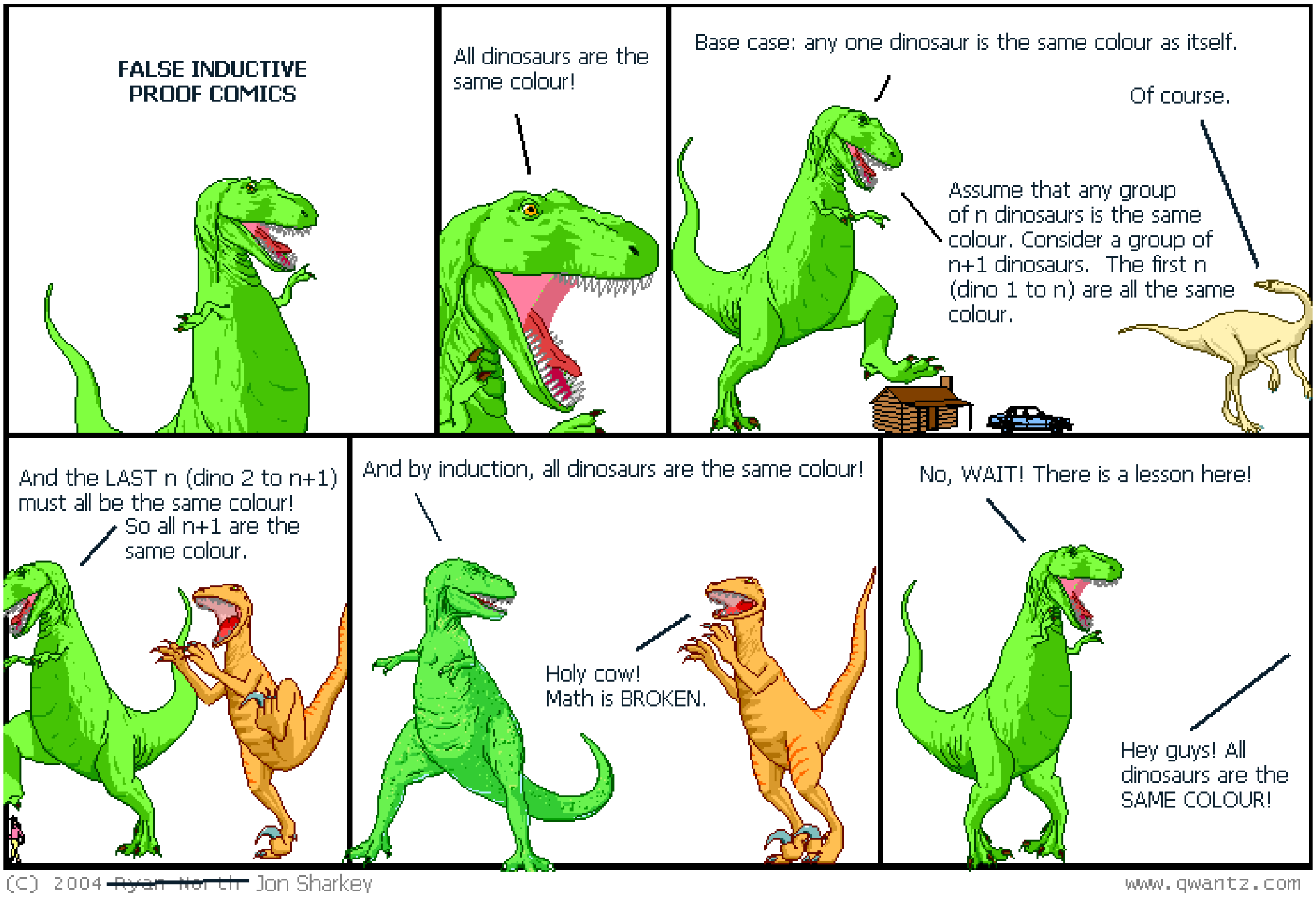
$3^{n+1} = 3^{n+1}$

9) Use numerical induction to prove that:
for all $n \in \mathbb{N}$: $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$

Base Case: $n = 0$
$\sum_{i=1}^{1} i(i!) =^? (1+1)! - 1$
$1(1!) =^? 2! - 1$
$1 = 1$

Induction Hypothesis: Assume that $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$
Induction Step: Show that this implies: $\sum_{i=1}^{n+1} i(i!) = ((n+1)+1)! - 1$
$\sum_{i=1}^{n+1} i(i!) =^? ((n+1)+1)! - 1$
$\sum_{i=1}^{n} i(i!) + (n+1)(n+1)! =^? ((n+1)+1)! - 1$
$(n+1)! - 1 + (n+1)(n+1)! =^? ((n+1)+1)! - 1$ Induction Hypothesis
$(n+1)! + (n+1)(n+1)! =^? ((n+1)+1)!$
$(n+1)!(1 + (n+1)) =^? ((n+1)+1)!$
$(n+1)!(n+2) =^? (n+2)!$
$(n+2)! = (n+2)!$

10) (Bonus) Explain carefully and clearly why math is not *actually* broken.



So this argument uses induction on the size of the group of dinosaurs. That's fine. It starts out with a base case of size one, showing that *all* groups of size one (all individual dinosaurs) are the same color as themselves. Then, there's an induction hypothesis... Assume that *all* groups of n dinosaurs are the same color. This is also fine, so long as we can show that this implies that *all* groups of n + 1 dinosaurs are the same color (the induction step). And the induction step itself is basically fine: Let's say we know that all groups of 5 dinosaurs are the same color, does his argument imply that all groups of 6 dinosaurs are the same color? Sure. Take a group of 6 dinosaurs. The first 5 are all the same color, by our induction hypothesis. And the group consisting of the second dinosaur through the sixth dinosaur is also a group of 5 dinos, so we know that *they're* all the same color too (since our hypothesis applied to *all* groups of size five). Since dinos 2 through 5 are in both groups, that implies that all 6 dinos are the same color.

ok.

So what's wrong? Try the argument on n=1 (n + 1 = 2).

Then, our induction hypothesis tells us that all groups of 1 dinosaur are the same color (all dinos are the same color as themselves). And the induction step tells us to look at dinos 1 through n (ie, the first dinosaur, since n = 1). It's the same color as itself. And then we look at dinos 2 through n+1 (ie, the second dino, since n = 1). And it's the same color as itself. But the intersection of *these* two groups is empty, so this doesn't do anything to tell us that the two dinosaurs are the same color. Thus, the whole thing pretty much breaks down.

Moral of the story, be careful that the argument you use for your induction step actually connects up correctly with your base case.