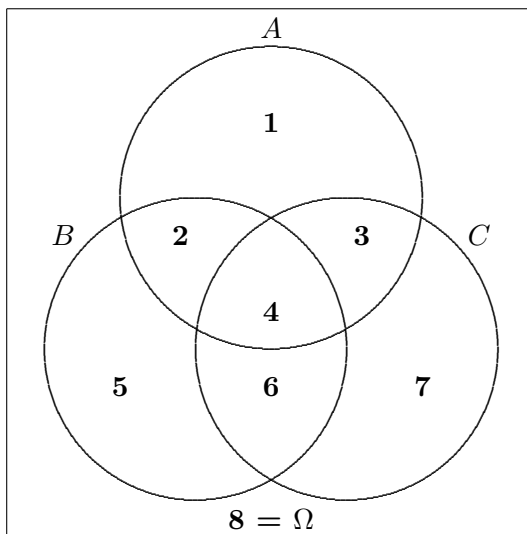


C241 Homework 4: Set Theory, Functions and Relations

Due: Wednesday 02/13/08

Venn Diagram for Three Sets



$$9 = \emptyset$$

1) Write the numbers of the Venn Diagram sections that you would shade to represent the following sets. Use 8 to indicate the *whole* space (or universe), Ω , and 9 to indicate the empty-set, \emptyset . For instance, set $B = 2, 4, 5, 6$ while set $A \cap B = 2, 4$.

a) $B \cap C \cap \bar{A}$

b) $A \cup B$

c) $A \cap B \cap C$

d) $A - B$

e) $A \cap \bar{A}$

f) $A \cup \bar{A}$

g) A

h) $A \cup (A \cap B)$

i) $A \cap (A \cup B)$

j) Use venn diagrams to prove: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2) Determine whether each of the following statements are True or False, and label them accordingly. If the statement is False, correct it so it's True. Remember " $A \Rightarrow B$ " does *not* mean the same thing as " $A \equiv B$ ". Inference rules, Laws of Logic, Set Theory definitions, and Quantifier equivalences are all applicable here (but you *do not* need to prove that your answers are correct.)

a) $(x \notin B) \equiv \neg(x \in B)$

b) $(B \subseteq A) \equiv \forall x[(x \in B) \rightarrow (x \in A)]$

c) $(B \subseteq A) \equiv \exists x[(x \in B) \rightarrow (x \in A)]$

d) $(B \subseteq A) \Rightarrow \exists x[(x \in B) \rightarrow (x \in A)]$

e) $((B \subseteq A) \vee (A \subseteq B)) \equiv (A = B)$

f) $((B \subseteq A) \wedge (A \subseteq B)) \equiv (A = B)$

g) $((B \subseteq A) \wedge (x \in B)) \Rightarrow (x \in A)$

h) $(B = \emptyset) \equiv (\exists x[x \in B])$

i) $\neg(\exists x[x \in B]) \equiv \forall x[x \notin B]$

$$\mathbf{j)} (x \in (A \cap B)) \equiv ((x \in A) \vee (x \in B))$$

$$\mathbf{k)} (x \in (A \cup B)) \equiv ((x \in A) \vee (x \in B))$$

$$\mathbf{l)} (x \notin (A \cup B)) \equiv ((x \notin A) \vee (x \notin B))$$

$$\mathbf{m)} (x \notin (A \cap B)) \equiv ((x \notin A) \wedge (x \notin B))$$

$$\mathbf{n)} (x \notin (A \cap B)) \equiv ((x \notin A) \vee (x \notin B))$$

$$\mathbf{o)} \forall x : (x \notin (A \cap B)) \equiv \forall x : ((x \in A) \rightarrow (x \notin B))$$

$$\mathbf{p)} \forall x[(x \in A) \wedge (x \in B)] \Rightarrow \forall x[(x \in A) \vee (x \in B)]$$

$$\mathbf{q)} \forall x[(x \in (A \cap B))] \equiv \forall x[(x \in (A \cup B))]$$

3) Use mutual containment to prove that: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, i.e., first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and then $A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$ without using the distributive laws.

4) Match each formal definition with its informal english equivalent.

- a) $\{a \in A \mid \exists b \in S \text{ such that } (a, b) \in F\}$
- b) $R \subseteq A \times B$
- c) $F : A \rightarrow B$ is a function, and $\forall b \in B, \exists a \in A$ such that $(a, b) \in F$
- d) $\{(b, a) \mid (a, b) \in F\}$
- e) $F \subseteq A \times B$ and $\forall a \in A, \exists$ *exactly one* $b \in B$ such that $(a, b) \in F$
- f) $\{b \in B \mid \exists a \in S \text{ such that } (a, b) \in F\}$
- g) $F : A \rightarrow B$ is a function, and $\forall b \in B, \exists$ *at most one* $a \in A$ such that $(a, b) \in F$

i) A **relation** $R : A \rightarrow B$ is a set of ordered pairs, where the first element in the pair comes from the set A, and the second element in the pair comes from the set B. We say the first element in each pair is "mapped" to the second element.

ii) A **total function** $F : A \rightarrow B$ is a relation in which every element in A is paired with *exactly one* element in B. This means that each element in A is paired with an element in B, and no element in A is paired with more than one element in B.

iii) The **image** of a set $S \subseteq A$ under a function $F : A \rightarrow B$ is the set of all elements in B that F pairs with the elements of A that are in S.

iv) The **pre-image** of a set $S \subseteq B$ under a function $F : A \rightarrow B$ is the set of all elements in A that F pairs with the elements of B that are in S.

v) An **injective function** $F : A \rightarrow B$ is a function in which no two elements of A are paired with the same element in B.

vi) A **surjective function** $F : A \rightarrow B$ is a function in which every element of B is paired with some element in A.

vii) F^{-1} , the **inverse** of a function $F : A \rightarrow B$, is the relation consisting of all the pairs in F flipped so that the second element is first, and the first element is second. The inverse of a function isn't always a function itself.

9) Prove that $(A \subseteq B) \equiv (A \cap \overline{B} = \emptyset)$ using logic. Start by replacing the statements $A \subseteq B$ and $A \cap \overline{B} = \emptyset$ with formal logical statements, using \exists , \forall , and \in . (Note that $A = \emptyset$ can be written as $\neg\exists x[x \in A]$), and then use set theory definitions and algebraic laws of logic to show that the statements are logically equivalent.

10) Label the following claims as True or False. If you think a claim is true, *explain why you think it's true*. If you think it's false, *give an example that shows that it is false*. (Note, when thinking about these problems and explaining your answers, it may help to draw graphs of the functions.)

a) If $F : A \rightarrow B$ is a surjective function, but it is *not* injective, then its inverse F^{-1} is also a function.

b) If $|A| < |B|$ (if set A has fewer elements than set B) then any function $F : A \rightarrow B$ is injective.

c) If $F : A \rightarrow B$ is a surjection, and there is another function $G : B \rightarrow A$ which is also a surjection, then you can find a third function $H : A \rightarrow B$ such that H is a bijection.