# C241 Assignment 2: Solutions

**1) Simplify the following complex statements using the Algebraic Laws of Logic (along with the equivalence $(p \rightarrow q) \equiv (\neg p \vee q)$ which you proved in the last assignment). Label each step you take with the justification for that step.**

Note: There was a typo in the original problem statement. $(p \rightarrow q)$ is equivalent to $(\neg p \vee q)$ but **not** $(\neg q \vee p)$.

**a)** $[(p \vee q) \wedge (p \vee \neg q)] \vee q$

| | |
|---|---|
| $\equiv [p \vee (q \wedge \neg q)] \vee q$ | First Distributive Law |
| $\equiv (p \vee \mathbf{F}) \vee q$ | Second Inverse Law |
| $\equiv p \vee q$ | First Identity |

**b)** $(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)]$

| | |
|---|---|
| $\equiv (p \rightarrow q) \wedge \neg q$ | Second Absorption Law |
| $\equiv (\neg p \vee q) \wedge \neg q$ | $(\neg q \vee p) \equiv (q \rightarrow p)$ |
| $\equiv \neg q \wedge (\neg p \vee q)$ | Second Commutative Law |
| $\equiv (\neg q \wedge \neg p) \vee (\neg q \wedge q)$ | Second Distributive Law |
| $\equiv (\neg q \wedge \neg p) \vee \mathbf{F}$ | Second Inverse Law |
| $\equiv (\neg q \wedge \neg p)$ | First Identity Law |
| $\equiv \neg(q \vee p)$ | First DeMorgan's Law |

**2) Answer the following questions clearly, in your own words:**

**a)** Given two compound statements A and B, describe two different methods for proving that A logically implies B (ie $A \Rightarrow B$).

* Write a truth table for each statement. If on each line of the truth table where statement A is True, statement B is True as well, then A logically implies B.
* Find a chain of Inference Rules leading from statement A to statement B.

**b)** If A and B are two compound statements and $A \Rightarrow B$, then what do we know about the compound statement $(A \rightarrow B)$?

Since whenever A is True, B is also True, we know $A \rightarrow B$ is a tautology.

**c)** Give an example of two compound statements A and B such that A logically implies B, but A and B are not logically equivalent (ie: $A \Rightarrow B$ but $A \not\equiv B$). Justify your choice using truth tables.

Choose A = $p \wedge q$ and B = $p \vee q$. Then their truth tables are:

| p q | $(p \wedge q)$ | $(p \vee q)$ |
|---|---|---|
| F F | F | F |
| F T | F | T |
| T F | F | T |
| T T | **T** | **T** |

A is only True when p and q are both True. At that line, B is also True, so this shows that A ⇒ B. However, the truth values for A and B do not match elsewhere, for instance if p is False and q is True, then B is True, but A is still False. So we know that A and B are not logically equivalent.

**d)** Can you give an example of an A and B such that A and B are logically equivalent, but A does not logically imply B? Why or why not?

You can't find such an A and B. If A ≡ B, then their truth tables match, which means that A ⇒ B and B ⇒ A.

**3) Label each step of the logical proof below with the justification for that step. Each of your labels should be one of the following: "hypothesis" (or "premise"), one of the logical equivalence laws from the chart at the end of this assignment sheet (Look carefully! There are a couple new laws for $(P \rightarrow Q)$ ), or a logical inference rule from the other chart at the end of this assignment sheet. Make sure you include the proof lines you're referencing in your label; the label for line (4) has been filled in as an example.**

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \neg u \rightarrow \neg t \\ \hline \therefore p \end{array}$$

| | | |
|---|---|---|
| 1) | $\neg s \wedge \neg u$ | premise |
| 2) | $\neg u$ | line (1) and Conjunctive Simplification |
| 3) | $\neg u \rightarrow \neg t$ | premise |
| 4) | $\neg t$ | lines (2)(3) and Modus Ponens |
| 5) | $\neg s$ | line (1) and Conjunctive Simplification |
| 6) | $\neg s \wedge \neg t$ | lines (4)(5) and Rule of Conjunction |
| 7) | $r \rightarrow (s \vee t)$ | premise |
| 8) | $\neg(s \vee t) \rightarrow \neg r$ | line (7) and $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$ (Contrapositive) |
| 9) | $(\neg s \wedge \neg t) \rightarrow \neg r$ | line (8) and DeMorgans |
| 10) | $\neg r$ | lines(6)(9) and Modus Ponens |
| 11) | $(\neg p \vee q) \rightarrow r$ | premise |
| 12) | $\neg r \rightarrow \neg(\neg p \vee q)$ | line (11) and $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$ (Contrapositive) |
| 13) | $\neg r \rightarrow (p \wedge \neg q)$ | line (12) and DeMoragans, and Double Negation |
| 14) | $p \wedge \neg q$ | lines (10)(13) and Modus Ponens |
| 15) | $\therefore p$ | line (14) and Conjunctive Simplification |

**4) Why was it acceptable to use logical equivalence laws in the proof above?**

We can use logical equivalences in these proofs because any two statements that are logically equivalent logically imply each other (see 1d).

**5) Do the following logical proofs on you own, using the same style as the proof in problem 2. There are many correct answers to these problems, the proofs below are one example.**

**a)**

$p \to q$
$\neg q$
$\neg r$

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$\therefore \neg(p \vee r)$

| | |
|---|---|
| 1.$p \to q$ | premise |
| 2.$\neg q$ | premise |
| 3.$\neg p$ | Modus Tollens (1)(2) |
| 4.$\neg p \wedge \neg q$ | Rule of Conjunction (2)(3) |
| 5.$\neg(p \vee q)$ | DeMorgan's (4) |

**b)** Do this as a proof by Resolution. First change each of the hypotheses to an 'or' statement, then apply the Rule of Proof by Resolution and simplify until you've completed the proof.

$\neg(\neg p \wedge \neg q)$
$p \to r$
$(\neg r)$

---

$\therefore q$

| | |
|---|---|
| 1.$\neg(\neg p \wedge \neg q)$ | premise |
| 2.$p \vee q$ | DeMorgan's (1) |
| 3.$p \to r$ | premise |
| 4.$\neg p \vee r$ | def. of Implication (3) |
| 5.$q \vee r$ | Proof by Resolution (4,2) |
| 6.$\neg r$ | premise |
| 7.$q$ | Rule of Disjunctive Syllogism (5,6) |

**c)** Do this as a proof by Contradiction. Add $\neg(s \vee t)$ to the list of hypotheses (i.e., assume the conclusion is false) and then continue the proof until you can derive 'False' as a conclusion (which shows that if $(s \vee t)$ *wasn't* true, then something False would be true, which is a contradiction).

$p$
$p \rightarrow q$
$s \vee r$
$r \rightarrow \neg q$

---

$\therefore s \vee t$

| | |
|---|---|
| 1,$\neg(s \vee t)$ | premise |
| 2.$\neg s \wedge \neg t$ | DeMorgans (1) |
| 3.$\neg s$ | Rule of Conjunctive Simplification (2) |
| 4.$s \vee r$ | premise |
| 5.$r$ | Rule of Disjunctive Syllogism (3,4) |
| 6. $r \rightarrow \neg q$ | premise |
| 7.$\neg q$ | Modus Ponens (5,6) |
| 8.$p \rightarrow q$ | premise |
| 9.$\neg p$ | Modus Tollens (7,8) |
| 10.$p$ | premise |
| 11.$\neg p \wedge p$ | Rule of Conjunction (9,10) |
| 12.$False$ | Contradiction (11) |

**d)**

$p \rightarrow q$
$\neg r \vee s$
$p \vee r$
$r \rightarrow \neg q$

---

$\therefore \neg q \rightarrow s$

| | |
|---|---|
| 1.$p \rightarrow q$ | premise |
| 2.$\neg r \vee s$ | premise |
| 3.$r \rightarrow s$ | definition of implication (2) |
| 4.$p \vee r$ | premise |
| 5.$q \vee s$ | Rule of the Constructive Dilemma (1,3,4) |
| 6. $\neg\neg q \vee s$ | double negation (5) |
| 7.$\neg q \rightarrow s$ | definition of implication (6) |

**6) Answer the following questions clearly, in your own words:**

**a)** What is an open statement?

An open statement is a logical statement with at least one free variable (ie, at least one variable which is not constrained by a quantifier). When you plug in values for each of the free variables in an open statement, the open statement becomes either True or False. Think of open statements as functions whose variables are their free variables, and whose range is {T,F}.

**b)** Write an open statement with two variables, and list the domain of each variable. For example, I could write: o(x,y) = "x is older than y", where the domain of both x and y is "cities".

q(x, y) = (x + y) <10, where x and y are integers.

**c)** Describe what the values of x and y need to be for your statement from part (b) to evaluate to true. Give an example of x,y values for which your statement evaluates to true. Give an example of x,y values for which your statement evaluates to false.

This open statement will be True when we plug in values for x and y such that x + y < 10. It will be False if we plug in values such that x + y ≥ 10. For example q(2,3) = True, while q(3, 11) = False.

**d)** What are the existential and universal quantifiers? (Give the symbols and english phrases for each).

∀ is the universal quantifier. It's read as "For all" or "For any" or "For every".

∃ is the existential quantifier. It's read as "There exists" or "For some" or "There is at least one"

**e)** Use a quantifier to bind the variable x in your statement from part b). For what values of y does this new statement evaluate to true?

Let's use the universal quantifier on x: $\forall x[x + y < 10]$. Remember that an open statement is a function whose variables are its free variables. Since we just bound x, it is no longer free–we can no longer plug in a specific value for x. Instead, this statement now makes a claim about *every* x in our domain, or in other words *every* integer. We could rename our open statement q(y) = $\forall x[x + y < 10]$, since the only free variable left is y. So the question here is, can we pick a value for y in our domain such that $\forall x[x + y < 10]$? Well, let's try -2. Is q(-2) = $\forall x[x + -2 < 10]$ True? Nope. It's False that "For *any* integer x, x + -2 is less than 10". Are we going to be able to find a value of y in our domain such that q(y) *is* True? Nope, we're not. For any value of y we could pick, there will be some integer x so big that x + y will still be greater than 10–thus the statement that, for *every* integer x, x+y < 10 will be false. Given that the domains for x and y are the integers, q(y) will always be false.

**f)** Use a quantifier to bind the remaining free variable in your statement from part e). When does this final statement evaluate to true? What can be said about a statement with no free variables?

Let's use the existential quantifier on y: $\forall x \exists y[x + y < 10]$. Remember that an open statement is a function whose variables are its free variables. We've now bound both x and y with quantifiers, so we don't have any free variables left! That means that this statement is now a function with no variables, or in other words it's a constant function. It will either be always True or always False. There are no variables left that we can plug values into to change the truth of the statement. So, which is it: True or False? Well, let's think about the claim it's making: "For every integer x, there is some integer y, such that x + y < 10." Well, if x is an integer, then (9 + -x ) is an integer too, and (x + (9 + -x)) = 9, which is less than 10. So it's True that, "For every integer x, there is some integer y (for example y = (9 + -x)), such that x + y < 10". Our final statement has a constant value of True.