C241 Assignment 3: Solutions

1) Below are examples of mistakes that you might find in a propositional logic proof (or simplification). Explain what's wrong with each.

a) $(\neg p \vee \neg q) \wedge r \equiv (p \wedge q \wedge r)$ **DeMorgan's, Associative**
DeMorgan's was applied incorrectly here. This is a good example of an occasion where writing each step out separate will help you avoid making mistakes. A corrected version:
$(\neg p \vee \neg q) \wedge r \equiv (\neg(p \wedge q) \wedge r)$ DeMorgan's

b) $((a \wedge b) \vee c) \equiv (a \wedge b \vee c)$ **Associative**
You can only use the Associative law if all of your connectives are the same (either all '$\wedge$' or all '$\vee$'). The following would be correct:
$((a \wedge b) \wedge c) \equiv (a \wedge b \wedge c)$ Associative
$((a \vee b) \vee c) \equiv (a \vee b \vee c)$ Associative
$((a \wedge b) \vee c) \equiv (a \vee b) \wedge (a \vee c)$ Distributive

c) $(\neg a \rightarrow b) \wedge (\neg b \rightarrow a) \equiv False$ **Contradiction**
You can only use negation laws (i.e., derive a contradiction) if you have something of the form $(P \wedge \neg P)$, where P is a single logical statement. This is another case where writing things out step by step can help. It's actually true that $(\neg a \rightarrow b) \equiv (\neg b \rightarrow a)$, so the following would be correct:
$(\neg a \rightarrow b) \wedge (\neg b \rightarrow a)$
$\equiv (\neg a \rightarrow b) \wedge (\neg a \rightarrow \neg\neg b)$ Contrapositive
$\equiv (\neg a \rightarrow b) \wedge (\neg a \rightarrow b)$ Double Negation
$\equiv (\neg a \rightarrow b)$ Idempotence

d) $(a \wedge (b \vee c)) \equiv (b \vee c)$ **Rule of Conjunctive Simplification**
This is a very important problem to understand. If we write $A \equiv B$, it means that the statement A is equivalent to the statement B. We can replace one with the other, because they have the same truth tables and the same meaning. Clearly, $(b \vee c)$ does *not* have the same truth table as $(a \wedge (b \vee c))$. *Instead*, it's true that $(a \wedge (b \vee c)) \Rightarrow (b \vee c)$: when $(a \wedge (b \vee c))$ is true, then $(b \vee c)$ is also true. We can use laws of logic in inference rule proofs, but we *cannot* use inference rules in simplifications or equivalence proofs. So it would be correct to say the following:
$(a \wedge (b \vee c)) \Rightarrow (b \vee c)$ Rule of Conjunctive Simplification

e) $(a \wedge (b \vee c)) \equiv (a \vee b) \wedge (a \vee c)$ **Distributive**
This was a misapplication of the distributive law. The correct statement would be:
$(a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$

f) $(a \vee \neg a) \equiv False$ **Contradiction**
$(a \wedge \neg a)$ would be a contradiction. "A or Not A", however, is always true... it's a tautology (since either A will be true or, if it's not, then $\neg A$ will be true). So to correct this we would write:
$(a \vee \neg a) \equiv True$ Inverse

2) Demonstrate the following logical equivalence by simplifying the statement on the left to the statement on the right

$(\neg p \vee (\neg p \wedge q)) \wedge (\neg p \wedge (\neg p \vee q)) \equiv \neg p$

$(\neg p \vee (\neg p \wedge q)) \wedge (\neg p \wedge (\neg p \vee q))$

$\equiv (\neg p \vee (\neg p \wedge q)) \wedge (\neg p)$ Subsumption

$\equiv (\neg p) \wedge (\neg p)$ Subsumption

$\equiv (\neg p)$ Idempotence

3)Simplify the following statements

a) $\neg(\forall x : p(x) \rightarrow q(x))$

$\equiv \exists x : \neg(p(x) \rightarrow q(x))$ negation of $\forall$

$\equiv \exists x : \neg(\neg p(x) \vee q(x))$ definition of $\rightarrow$

$\equiv \exists x : (\neg\neg p(x) \wedge \neg q(x))$ DeMorgan's

$\equiv \exists x : (p(x) \wedge \neg q(x))$ Double Negation

b) $\neg(\forall x \forall y : (x < y) \rightarrow \exists z : (x < z < y))$

$\equiv \exists x \neg\forall y : (x < y) \rightarrow \exists z : (x < z < y)$ Negation of $\forall$

$\equiv \exists x \exists y : \neg((x < y) \rightarrow \exists z : (x < z < y))$ Negation of $\forall$

$\equiv \exists x \exists y : \neg(\neg(x < y) \vee \exists z : (x < z < y)))$ Definition of $\rightarrow$

$\equiv \exists x \exists y : (\neg\neg(x < y) \wedge \neg\exists z : (x < z < y))$ DeMorgan's

$\equiv \exists x \exists y : ((x < y) \wedge \neg\exists z : (x < z < y))$ Double Negation

$\equiv \exists x \exists y : ((x < y) \wedge \forall z : \neg(x < z < y))$ Negation of $\exists$

This above is a fine stopping point. However, if you did choose to go further, this is how you'd do it:

$\equiv \exists x \exists y : ((x < y) \wedge \forall z : \neg((x < z) \wedge (z < y)))$

$\equiv \exists x \exists y : ((x < y) \wedge \forall z : (\neg(x < z) \vee \neg(z < y)))$ DeMorgans

$\equiv \exists x \exists y : ((x < y) \wedge \forall z : ((x \geq z) \vee (z \geq y)))$

4) Answer each question below as "yes (Y), "no" (N), or "maybe"(M). Think *very* carefully about your answer, you should use 'maybe' frequently.

The domain of x is students.

$P(x) =$"x has a pie", $A(x) =$"x has an apple pie"

a) If it's true that: $\exists x : P(x)$, then:

(M) i) Do all students have pies?

(If all students did have pies, then we can find an example of a student with a pie. On the other hand, that would also be true if only one student had a pie).

(Y) ii) Does some student have a pie?

(This statement tells us that we can find an example of a student with a pie).

b) If it's true that: $\forall x : A(x)$, then:

(Y) i) Does some student have an apple pie?

(This statement tells us that all students have apple pies, so certainly we can find an example of a student with an apple pie).

(Y) ii) Do all students have apple pies?

(Yes, that's what this statement means)

(M) iii) If a pie belongs to a student, is it an apple pie?

(All students have apple pies, but a student could have a strawberry pie in addition to his apple pie.)

c) If it's true that: $\forall x : \neg A(x)$, then:

(N) i) Does some student have an apple pie?

(This statement tells that all students don't have apple pies, or equivalently, that we can't find an example of a student with an apple pie: $\neg\exists x : A(x)$).

(M) ii) Does some student have a pie?

(Just because no one has an apple pie, doesn't mean someone couldn't have a pecan pie).

d) If it's true that: $\neg\forall x : A(x)$, then:

(N) i) Does every student have an apple pie?

(This statement tells us that not everyone has an apple pie. That means at least someone doesn't have an apple pie (if we move the negation inside): $\exists x : \neg A(x)$).

(M) ii) Does any student have an apple pie?

(It's possible. All this statement says is that at least one student doesn't have an apple pie. It could be that no one has an apple pie, or it could be that everyone but bob has an apple pie. We don't know).

e) If it's true that: $\neg\exists x : P(x)$, then:

(N) i) Does every student have a pie?

(This statement says that we can't find an example of a student with a pie. Thus *no one* has a pie).

(N) ii) Does any student have a pie?

(Nope. No one has a pie).

f) If it's true that: $\forall x : P(x) \wedge A(x)$ **then:**

(N) i) Is there any student who doesn't have a pie?

(This statement says that all students have pies and all students have apple pies. Thus we know that all students have pies).

(M) ii) If a pie belongs to a student, is it an apple pie?

(We know that all students have pies, and that all students have apple pies. But a student *could* have both an apple pie and a banana creme pie. Just because everyone has an apple pie doesn't mean that all pies are apple).

(M) iii) Does every student have two pies?

(Maybe. But all we know from this statement is that everyone has at least one apple pie.)

g) If it's true that: $\forall x : P(x) \rightarrow A(x)$ **then:**

(M)i) Do all students have pies?

(This statement says that, for all students, *if* a student has a pie, then that student has an apple pie. However, it doesn't claim that any student actually *does* have a pie, it just says what would be true if a student has a pie.)

(M) ii) Does some student have a pie?

(It's actually possible that no one has a pie)

(Y) iii) If a student has only one pie, is it an apple pie?

(Yes, if a student has a pie, then that student has an apple pie. And if the student only has *one* pie, then that pie must *be* the apple pie.)

h) If it's true that: $\exists x : P(x) \vee A(x)$ **then:**

(Y) i) Does some student have a pie?

(This statement says that we can find an example of a student who either has a pie or has an apple pie. So at least someone out there has a pie).

(M) ii) Does some student have an apple pie?

(If there was only one student, with a single blueberry pie, this statement would still be true... since that student would be an example of an x that satisfied $P(x)$, and if $P(x)$ is true, then $P(x) \vee A(x)$ is true).

(M) iii) Do all student have pies?

(If everyone had a pie, then we could easily find an example of a student with a pie).

5) Label each step of the logical proof below with the justification for that step.

$$\begin{array}{l} \forall x : (p(x) \lor q(x)) \\ \exists x : \neg p(x) \\ \forall x : (\neg q(x) \lor r(x)) \\ \forall x : (s(x) \rightarrow \neg r(x)) \\ \hline \therefore \exists x : \neg s(x) \end{array}$$

1)	$\forall x : (p(x) \lor q(x))$	premise
2)	$\exists x : \neg p(x)$	premise
3)	$\neg p(c)$	Existential Specification (2)
4)	$p(c) \lor q(c)$	Universal Specification (1)(3) (since $(p(x) \lor q(x)$ is true for all x, we know it's true for the specific case c)
5)	$q(c)$	Disjunctive Syllogism (3)(4)
6)	$\forall x : (\neg q(x) \lor r(x))$	premise
7)	$(\neg q(c) \lor r(c))$	Universal Specification (6)
8)	$(q(c) \rightarrow r(c))$	Definition of $\rightarrow$ (7)
9)	$r(c)$	Modus Ponens (8)(5)
10)	$\forall x : s(x) \rightarrow \neg r(x)$	premise
11)	$(s(c) \rightarrow \neg r(c))$	Universal Specification (10)
12)	$(\neg\neg r(c) \rightarrow \neg s(c))$	Contrapositive (11)
13)	$(r(c) \rightarrow \neg s(c))$	Double Negation (12)
14)	$\neg s(c)$	Modus Ponens (9)(13)
15)	$\therefore \exists x : \neg s(x)$	Existential Generalization

6) Do the following two proofs, in the same style as was used above (label each step).

a)
$$\begin{array}{l} \forall x : \neg r(x) \\ \forall x : (p(x) \vee q(x)) \\ \forall x : (\neg p(x) \wedge q(x)) \rightarrow r(x) \\ \hline \therefore \forall x : (\neg r(x) \wedge p(x)) \end{array}$$

1)	$\forall x : \neg r(x)$	premise
2)	$\neg r(a)$	Universal Specification(1) ('a' here represents any arbitrarily chosen value for x)
3)	$\forall x : (p(x) \vee q(x))$	premise
4)	$(p(a) \vee q(a))$	Universal Specification(3) (this is the same 'a' from 2)
5)	$\forall x : (\neg p(x) \wedge q(x)) \rightarrow r(x)$	premise
6)	$(\neg p(a) \wedge q(a)) \rightarrow r(a)$	Universal Specification (5) (again the same 'a' from 2)
7)	$\neg(\neg p(a) \wedge q(a))$	Modus Tollens (2)(6)
8)	$(\neg\neg p(a) \vee \neg q(a))$	DeMorgans (7)
9)	$(p(a) \vee negq(a))$	Double Negation (8)
10)	$(p(a) \vee p(a))$	Proof by Resolution (9)(4)
11)	$p(a)$	Idempotence (10)
12)	$\neg r(a) \wedge p(a))$	Rule of Conjunction (11)(2)
13)	$\forall x : (\neg r(x) \wedge p(x))$	Universal Generalization (12) (since we showed this was true for any arbitrarily chosen x it's true for all x).

b)
$$\begin{array}{l} \exists x : \neg t(x) \\ \forall x : (\neg p(x) \vee \neg q(x)) \rightarrow (r(x) \wedge s(x)) \\ \forall x : (r(x) \rightarrow t(x)) \\ \hline \therefore \exists x : p(x) \end{array}$$

1)	$\exists x : \neg t(x)$	premise
2)	$\neg t(c)$	Existential Specification. (1) (We know there's at least one value for x such that $\neg t(x)$ is true. Let's pick one of those values and call it 'c')
3)	$\forall x : (r(x) \rightarrow t(x))$	premise
4)	$(r(c) \rightarrow t(c))$	Universal Specification(3) (Since this is true for every value of x, we know it's true for $x = c$.)
5)	$\neg r(c)$	Modus Tollens (4)(2)
6)	$\neg r(c) \vee \neg s(c)$	Disjunctive Amplification (5)
7)	$\neg(r(c) \wedge s(c))$	DeMorgan's (6)
8)	$\forall x : (\neg p(x) \vee \neg q(x)) \rightarrow (r(x) \wedge s(x))$	premise
9)	$(\neg p(c) \vee \neg q(c)) \rightarrow (r(c) \wedge s(c))$	Universal Specification (8)
10)	$\neg(\neg p(c) \vee \neg q(c))$	Modus Tollens (7)(9)
11)	$(\neg\neg p(c) \wedge \neg\neg q(c))$	DeMorgans (10)
12)	$(p(c) \wedge q(c))$	Double Negation (11)
13)	$p(c)$	Conjunctive Simplification (12)
14)	$\exists x : p(x)$	Existential Generalization (13) (we know this is true for $x = c$, which means we can find an example of an x where this is true.)

7) Do the following proofs by contradiction. *You do not need to use formal logic notation,* **but ideas like Universal Specification and Generalization will be useful. Explain your reasoning carefully. Remember that if an integer n is odd, then $n = 2m+1$ for some integer m. And if n is even, then $n = 2m$ for some integer m.**

a) Prove that for any integer n, if n is odd, then $n + 11$ is even.

Let us denote the statement "n is odd" with P and "$n + 11$ is even" with Q. Since we're doing this by contradiction, we'll start out by assuming that P is true (i.e., "n is odd") but that Q is false (i.e., "$n + 11$ is odd"). That is, we assume $P \wedge \neg Q$ and try to derive a contradiction (false!). Since we assumed n is odd we can write it as $n = 2m + 1$ for some integer m. But then we can write $n + 11$ as $2m + 1 + 11$ (i.e., we are "plugging in" the formula). Now, $2m + 1 + 11 = 2m + 12 = 2(m + 6)$. Hence, since every integer which can be written as $2k$ for some integer k is even, we can conclude that $n + 11$ must be even. But this is a contradiction to our assumption $\neg Q$, i.e., "n+11 is odd". Thus we've found a contradiction, and our original assumption must have been wrong. We have shown that if n is odd, then $n + 11$ must be even.

b) Prove that for every integer n, if n^2 is odd, then n is odd.

We will show that the contrapositive of the statement is true and thus that the original statement is true. Let us denote the statement "n^2 is odd" with P and "n is odd" with Q. Hence, the statement we want to prove is $P \Rightarrow Q$. Instead of proving the statement directly, we can prove that the contrapositive of the statement is true (i.e., an indirect prrof): $\neg Q \Rightarrow \neg P$.

Since we want to proof the contrapositive of the statement by contradiction, we'll start out by assuming that $\neg Q$ is true (i.e., "n is even") but that P is true (i.e., "n^2 is odd"). That is, we assume $\neg Q \wedge P$ and try to derive a contradiction (false!). Since we assumed n is even we can write it as $n = 2m$ for some integer m. But then we can write n^2 as $(2m)^2$ (i.e., we are "plugging in" the formula). Now, $(2m)^2 = 4m^2 = 2(2m^2)$. Since every integer which can be written as $2k$ for some integer k is even, we can conclude that n^2 must be even. But this is a contradiction to our assumption that P is true, i.e., "n^2 is odd". We've found a contradiction, and our assumption must have been wrong. Hence, we have shown that if n is even, then n^2 must be even as well. But that is (by contrapositive) equivalent to "If n^2 is odd, then n is odd".

8) List the elements of the following sets.

a) $\{1,2,3,8\} \cap \{2,8\} = \{2,8\}$

b) $\{1,2,3,8\} \cup \{2,8,5\} = \{1,2,3,8,5\}$

c) $\{1,2,3,8\} - \{2,8\} = \{1,3\}$

d) $\{1,2,3,8\} \cap \{5,7\} = \emptyset$

e) $\{1,2,3,8\} \cap \emptyset = \emptyset$

f) $\{1,2,3,8\} \cup \emptyset = \{1,2,3,8\}$

g) power set of $\{2,8\} = \{\emptyset, \{2\}, \{8\}, \{2,8\}\}$

h) power set of $\emptyset = \{\emptyset\}$

9) Label each of these statemens as either True or False.:

a) $\{2,8\} \subseteq \{1,2,3,8\}$: T

b) $\{2,8\} \in \{1,3,\{2,8\}\}$: T

c) $\{2,8\} \subseteq \{1,3,\{2,8\}\}$: F

d) $\{1,3\} \in \{1,3,\{2,8\}\}$: F

e) $\emptyset \in \{1,3,2,8,\emptyset\}$: T

f) $\emptyset \in \{1,3,2,8\}$: F

g) $\emptyset \subseteq \{1,3,2,8\}$: T

h) $\{1,2,3,8\} \subseteq \{1,2,3,8\}$: T

i) $\{1,2,3,8\} \subset \{1,2,3,8\}$: F

j) $\{\emptyset\} = \emptyset$: F

k) $\emptyset \subseteq \{\emptyset\}$: T