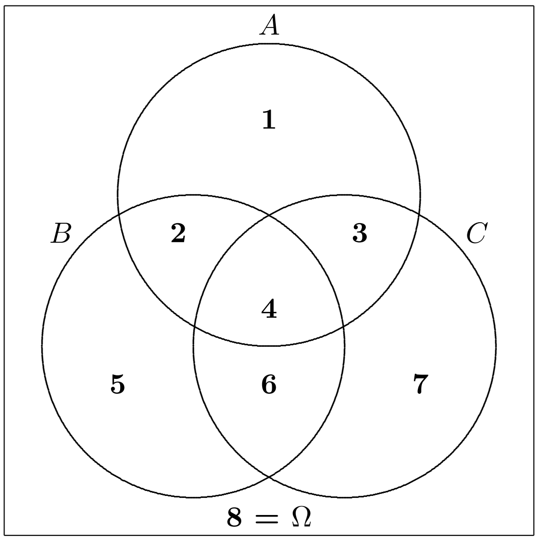
C241 Homework 4: Solutions

Venn Diagram for Three Sets



9 $= \emptyset$

1) Write the numbers of the Venn Diagram sections that you would shade to represent the following sets. Use 8 to indicate the *whole* space (or universe), Ω, and 9 to indicate the empty-set, $\emptyset$. For instance, set B = 2, 4, 5, 6 while set $A \cap B$ = 2, 4.

a) $B \cap C \cap \overline{A} = 6$

b) $A \cup B = 1, 2, 3, 4, 5, 6$

c) $A \cap B \cap C = 4$

d) $A - B = 1, 3$

e) $A \cap \overline{A} = 9$

f) $A \cup \overline{A} = 8$

g) $A = 1, 2, 3, 4$

h) $A \cup (A \cap B) = 1, 2, 3, 4$

i) $A \cap (A \cup B) = 1, 2, 3, 4$

j) Use venn diagrams to prove: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2) Determine whether each of the following statements are True or False, and label them accordingly. If the statement is False, correct it so it's True. Remember "$A \Rightarrow B$" does *not* mean the same thing as "$A \Leftrightarrow B$". Inference rules, Laws of Logic, Set Theory definitions, and Quantifier equivalences are all applicable here (but you *do not* need to prove that your answers are correct.)

a) $(x \notin B) \Leftrightarrow \neg(x \in B)$

T

b) $(B \subseteq A) \Leftrightarrow \forall x[(x \in B) \rightarrow (x \in A)]$

T

c) $(B \subseteq A) \Leftrightarrow \exists x[(x \in B) \rightarrow (x \in A)]$

F: $(B \subseteq A) \Rightarrow \exists x[(x \in B) \rightarrow (x \in A)]$

d) $(B \subseteq A) \Rightarrow \exists x[(x \in B) \rightarrow (x \in A)]$

T

e) $((B \subseteq A) \vee (A \subseteq B)) \Leftrightarrow (A = B)$

F: $((B \subseteq A) \wedge (A \subseteq B)) \Leftrightarrow (A = B)$

f) $((B \subseteq A) \wedge (A \subseteq B)) \Leftrightarrow (A = B)$

T

g) $((B \subseteq A) \wedge (x \in B)) \Rightarrow (x \in A)$

T

h) $(B = \emptyset) \Leftrightarrow (\exists x[x \in B])$

F: $(B = \emptyset) \Leftrightarrow \neg(\exists x[x \in B])$

i) $\neg(\exists x[x \in B]) \Leftrightarrow \forall x[x \notin B]$

T

j) $(x \in (A \cap B)) \Leftrightarrow ((x \in A) \vee (x \in B))$
F: $(x \in (A \cap B)) \Leftrightarrow ((x \in A) \wedge (x \in B))$

k) $(x \in (A \cup B)) \Leftrightarrow ((x \in A) \vee (x \in B))$
T

l) $(x \notin (A \cup B)) \Leftrightarrow ((x \notin A) \vee (x \notin B))$
F: $(x \notin (A \cup B)) \Leftrightarrow ((x \notin A) \wedge (x \notin B))$

m) $(x \notin (A \cup B)) \Leftrightarrow ((x \notin A) \wedge (x \notin B))$
T

n) $(x \notin (A \cap B)) \Leftrightarrow ((x \notin A) \vee (x \notin B))$
T

o) $\forall x : (x \notin (A \cap B)) \Leftrightarrow \forall x : ((x \in A) \rightarrow (x \notin B))$
T

p) $\forall x[(x \in A) \wedge (x \in B)] \Rightarrow \forall x[(x \in A) \vee (x \in B)]]$
T

q) $\forall x[(x \in (A \cap B))] \Leftrightarrow \forall x[(x \in (A \cup B))]$
F: $\forall x[(x \in (A \cap B))] \Rightarrow \forall x[(x \in (A \cup B))]$

3) Use mutual containment to prove that: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

You need to prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. There's a few different ways to do this. For example, you could use logic or a proof by cases (which is frequently the way you need to do mutual containment proofs).

Logic:
To show: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$
$\forall x : [(x \in (A \cap (B \cup C))) \Rightarrow (x \in ((A \cap B) \cup (A \cap C)))]$ Definition of subset
$[a \in (A \cap (B \cup C))] \Rightarrow [a \in ((A \cap B) \cup (A \cap C))]$ Universal Specification
$[(a \in A) \wedge ((a \in B) \vee (a \in C))] \Rightarrow [((a \in A) \wedge (a \in B)) \vee ((a \in A) \wedge (a \in C))]$
Definitions of union and intersection
The above is true by the distributive law **of logic**. Since the distributive law **of logic** is a logical equivalence, it's also true that $[(a \in A) \wedge ((a \in B) \vee (a \in C))] \Leftarrow [((a \in A) \wedge (a \in B)) \vee ((a \in A) \wedge (a \in C))]$. Thus the second part of the proof, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ will be similar. Here, you have shown the distributive law **for sets** by reducing it to and using the distributive law **of logic**.

By Cases:
To show: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
If an arbitrary element x is in $A \cap (B \cup C)$ then $x \in A \wedge ((x \in B) \vee (x \in C))$.
Hence, we know $x \in A$ and we have three different cases that make $((x \in B) \vee (x \in C))$ evaluate to true.

(1) $x \in B \wedge x \notin C$: Then x is in $(A \cap B)$ and therefore in $(A \cap B) \cup (A \cap C)$ by definition of union.
(2) $x \notin B \wedge x \in C$: Then x is in $(A \cap C)$ and therefore in $(A \cap B) \cup (A \cap C)$ by definition of union.
(3) $x \in B \wedge x \in C$: Then x is in $(A \cap B)$ *and* x is in $(A \cap C)$, and therefore, of course, also in $(A \cap B) \cup (A \cap C)$ by definition of union.

Note that the only possible case left, viz. $x \notin B \wedge x \notin C$, is *not* possible, because in this case $((x \in B) \vee (x \in C))$ would evaluate to false. Thus, we have shown that if an arbitrary element x is an element of $A \cap (B \cup C)$ then it is also an element of $(A \cap B) \cup (A \cap C)$, and hence, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
The second part of the proof, i.e., the other "direction" of the containment, works analogous.

4) Match each formal definition with its informal english equivalent.

i) A ***relation*** $R : A \to B$ is a set of ordered pairs, where the first element in the pair comes from the set A, and the second element in the pair comes from the set B. We say the first element in each pair is "mapped" to the second element.

b) $R \subseteq A \times B$

ii) A ***function*** $F : A \to B$ is a relation in which every element in A is paired with *exactly one* element in B. This means that each element in A is paired with an element in B, and no element in A is paired with more than one element in B.

e) $F \subseteq A \times B$ and $\forall a \in A, \exists$ *exactly one* $b \in B$ such that $(a, b) \in F$

iii) The ***image*** of a set $S \subseteq A$ under a function $F : A \to B$ is the set of all elements in B that F pairs with the elements of A that are in S.

f) $\{b \in B | \exists a \in S \text{ such that } (a, b) \in F\}$

iv) The ***pre-image*** of a set $S \subseteq B$ under a function $F : A \to B$ is the set of all elements in A that F pairs with the elements of B that are in S.

a) $\{a \in A | \exists b \in S \text{ such that } (a, b) \in F\}$

v) An ***injective function*** $F : A \to B$ is a function in which no two elements of A are paired with the same element in B.

g) $F : A \to B$ is a function, and $\forall b \in B$, $\exists$ *at most* one $a \in A$ such that $(a, b) \in F$

vi) A ***surjectve function*** $F : A \to B$ is a function in which every element of B is paired with some element in A.

c) $F : A \to B$ is a function, and $\forall b \in B$, $\exists a \in A$ such that $(a, b) \in F$

vii) F^{-1}, the ***inverse*** of a function $F : A \to B$, is the relation consisting of all the pairs in F flipped so that the second element is first, and the first element is second. The inverse of a function isn't always a function itself.

d) $\{(b, a) | (a, b) \in F\}$

5) Using the sets A = {1, 2, 3, 4} and B = {a, b, c, d}, give examples for each of the terms that were defined in problem 6 above.

a) relation {(1, a), (3, c), (3, d)}

b) function {(1, b), (3, d), (2, c), (4, c)}

c) image (use your F from part b) S = {2, 4} Image of S = {c}

d) preimage (use your F from part b) T = {b, c} Pre-image of T = {1, 2}

e) injective function {(1, a), (2, b), (3, d), (4, c)}

f) surjective function {(1, a) (2, b), (3, d), (4, c)}

g) bijective function (a function that's both injective and surjective)
{(1, a) (2, b), (3, d), (4, c)}

h) inverse (use your F from part g) {(a, 1), (b, 2), (d, 3), (c, 4)}

6) Let A = {1, 2} and B = {2, 3, 4}. Which of the following relations from A to B are total functions?

a) { (1, 3), (2, 4) } × c) { (1, 3), (1, 3) } × e) { (1, 3), (2, 5) }
× b) { (1, 3), (1, 4) } **d) { (2, 2), (1, 4) }**

(On e) note that $5 \notin B$)

7) Given a total function F = {(1, 2), (2, 3)}, which of the following are a *valid* domain and co-domain for F?

x a) F: $\mathbb{N} \to \mathbb{N}$ x d) F: {(1, 2)} → {(2, 3)}
b) F: {1, 2} $\to \mathbb{N}$ x e) F: {1, 2, 3} → {2, 3}
x c) F: {1, 2, 3} → {1, 2, 3}

8) Label the following functions as: injective, surjective, both (bijective), or neither.

a) $F : \{1,2,3\} \to \{a,b\}, F = \{(1,a),(2,b),(3,b)\}$ surjective

b) $F : \{1,2\} \to \{a,b,c\}, F = \{(1,a),(2,b)\}$ injective

c) $F : \{1,2,3\} \to \{a,b,c\}, F = \{(1,a),(3,b),(2,c)\}$ bijective

d) $F : \{1,2,3\} \to \{a,b,c\}, F = \{(1,b),(2,b),(3,a)\}$ neither

e) $F : \{1,2\} \to \{a,b,c\}, F = \{(1,a),(2,a)\}$ neither

9) Prove that $(A \subseteq B) \Leftrightarrow (A \cap \overline{B} = \emptyset)$ using logic. Start by replacing the statements $A \subseteq B$ and $A \cap \overline{B} = \emptyset$ with formal logical statements, using $\exists$, $\forall$, and $\in$. (Note that $A = \emptyset$ can be written as $\neg\exists x[x \in A]$), and then use set theory definitions and algebraic laws of logic to show that the statements are logically equivalent.

$A \subseteq B$ can be written as $\forall x[(x \in A) \to (x \in B)]$

$A \cap \overline{B} = \emptyset$ can be written as $\neg\exists x[x \in (A \cap \overline{B})]$

$\neg\exists x[x \in (A \cap \overline{B})] \Leftrightarrow$	
$\forall x \neg[x \in (A \cap \overline{B})] \Leftrightarrow$	Quantifier negation rules
$\forall x \neg[(x \in A) \wedge (x \in \overline{B})] \Leftrightarrow$	Definition of Intersection
$\forall x \neg[(x \in A) \wedge \neg(x \in B)] \Leftrightarrow$	Definition of Complement
$\forall x[\neg(x \in A) \vee \neg\neg(x \in B)] \Leftrightarrow$	DeMorgan's
$\forall x[\neg(x \in A) \vee (x \in B)] \Leftrightarrow$	Double Negation
$\forall x[(x \in A) \to (x \in B)] \Leftrightarrow$	$(\neg p \vee q) \Leftrightarrow (p \to q)$
$A \subseteq B$	Definition of subset

10) Label the following claims as True or False. If you think a claim is true, *explain why you think it's true.* If you think it's false, *give an example that shows that it is false.* (Note, when thinking about these problems and explaining your answers, it may help to draw graphs of the functions.)

a) If $F : A \to B$ is a surjective function, but it is *not* injective, then its inverse F^{-1} is also a function.

This is false. In fact, if F isn't injective, its inverse can *never* be a function. See what happens with $A = \{1,2,3\}, B = \{a,b\}$, and $F = \{(1,a)(2,b),(3,b)\}$. Note that this isn't injective because two elements of A are paired with the same element of B. When we look at the inverse we see $F^{-1} = \{(a,1),(b,2),(b,3)\}$. Now one element in the domain, b, is paired with two different elements in the range. Since a function is a relation that pairs each element in the domain with *exactly one* element in the range, this means that F^{-1} is not a function.

b) If $|A| < |B|$ (if set A has fewer elements than set B) then any function $F : A \to B$ is injective.

This is false... just because it's possible to have an injective function between these sets does not mean that *every* function between them has to be injective. For instance, consider $A = \{1,2,3\}, B = \{a,b,c,d\}$, and the function $R = \{(1,a),(2,a),(3,a),(4,a)\}$. Set B is larger than set A, but that function is definitely not injective.

c) If $F : A \to B$ is a surjection, and there is another function $G : B \to A$ which is also a surjection, then you can find a third function $H : A \to B$ such that H is a bijection.

This is true. So, up above on d) we showed if $F : A \to B$ is a surjection, then $|A|$ can't be less than $|B|$. This means that if F is a surjection, then $|A| \geq |B|$. And since $G : B \to A$ is also a surjection, we know that $|B| \geq |A|$. If it's true that $|A| \geq |B|$ *and* $|B| \geq |A|$, then it must be true that $|B| = |A|$. And if two sets have the same number of elements, then it's easy to make an bijective function between them: Each element of A can be paired with a different element of B (so it's injective), and once you've done that, because B has the same number of elements as A, each element of B will have been paired with an element in A (so it's surjective).