C241 Assignment 9: Solutions

1) For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

(a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$. **transitive**

(b) $\{(2, 4), (4, 2)\}$. **symmetric**

(c) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$. **reflexive, symmetric, anti-symmetric, transitive**

(d) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$. **none of the above properties**

2) Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

(a) $x \neq y$. **symmetric**

(b) $xy \geq 1$. **symmetric, transitive** (not reflexive because $00 \ngeq 1$)

(c) $x = y + 1$ or $x = y - 1$. **symmetric**

(d) x is a multiple of y. **reflexive, anti-symmetric, transitive**

(e) x and y are both negative or both non-negative. **reflexive, symmetric, transitive**

(f) $x = y^2$. **anti-symmetric**

(g) $x \geq y^2$. **anti-symmetric, transitive**

3) Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n.

Solution: We prove this by induction:
Base case: $R^1 = R$. This is obviously true.

Induction hypothesis: $R^k = R$.
To show: $R^{k+1} = R$. Since $R^{k+1} = R^k \cdot R$ we can use the IH and conclude that we only need to show that $R^2 = R \cdot R = R$.

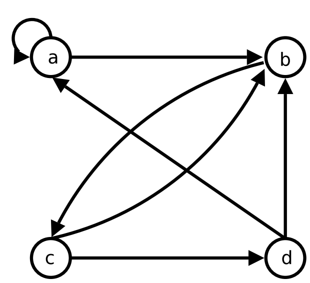
$R \cdot R$ is the composition of the relation R with itself. We prove that $R \cdot R = R$ by contradiction. Let $(a, c) \in R \cdot R$ and assume $(a, c) \notin R$. But since $(a, c) \notin R$ the ordered pairs (a, b) and (b, c) cannot be in R because otherwise, by transitivity, $(a, c) \in R$. Thus, (a, c) cannot be in $R \cdot R$. A contradiction to our assumption that $(a, c) \in R \cdot R$.
Now, assume that $(a, c) \in R$ but $(a, c) \notin R \cdot R$. But this leads immediatly to a contradiction, because since R is reflexive, we have that $(a, a) \in R$ and then by the definition of the composition of two relations, $(a, c) \in R \cdot R$.

4) List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these (incidence) matrices (where the rows and columns correspond to the integers listed in increasing order).

(a) $\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix}$ **Solution:** $\{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)\}$

(b) $\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix}$ **Solution:** $\{(1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4), (4, 1), (4, 3)\}$

5) Draw the directed graph that represents the relation
$\{(a, a), (a, b), (b, c)(c, b), (c, d), (d, a), (d, b)\}$**.**



6) Let R be the relation represented by the matrix $\mathbf{M}_R = \begin{vmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{vmatrix}$

(a) **Solution:** $R \cdot R = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = R^2.$

(b) **Solution:** $R^3 = R^2 \cdot R =$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

7) Which of these relations on $\{0, 1, 2, 3\}$ are quivalence relations? Determine the properties of an equivalence relation that the others lack. For those relations that are equivalence relations, what are the equivalence classes?

(a) $\{(0,0),(1,1),(2,2),(3,3)\}$ Equivalence classes: $\{0\},\{1\},\{2\},\{3\}$

(b) $\{(0,0),(0,2),(2,0),(2,2),(2,3),(3,2),(3,3)\}$ not reflexive

(c) $\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)\}$ Equivalence classes: $\{0\},\{1,2\},\{3\}$

(d) $\{(0,0),(1,1),(1,3),(2,2),(2,3),(3,1),(3,2),(3,3)\}$ not transitive

(e) $\{(0,0),(0,1)(0,2),(1,0),(1,1),(1,2),(2,0),(2,2),(3,3)\}$ not symmetric

8) Show that propositional equivalence is an equivalence relation on the set of all compound propositions.

For all propositional compound statements we have that

reflexivity: $p \equiv p$, for all compound statements p,

symmetry: $(p \equiv q) \Rightarrow (q \equiv p)$, and

transitivity: $(p \equiv q)$ and $(q \equiv r) \Rightarrow (p \equiv r)$

9) Show that the relation R on the set of all bit strings such that $s\ R\ t$ (or, in different notation, $(s,t) \in R$) if and only if s and t contain the same number of 1s is an equivalence relation.

For all bitstrings we have that

> **reflexivity**: num_of_ones(s) = num_of_ones(s), for all bitstrings s,
>
> **symmetry**: num_of_ones(s) = num_of_ones(t) ⇒ num_of_ones(t) = num_of_ones(s), for all bitstrings s, t; and
>
> **transitivity**: num_of_ones(s) = num_of_ones(t) and num_of_ones(t) = num_of_ones(u) ⇒ num_of_ones(s) = num_of_ones(u), for all bitstrings s, t, u.

10) Which of these are posets?

(a) $(\mathbf{Z}, =)$ **poset**

(b) $(\mathbf{Z}, \neq)$ **not a poset**

(c) $(\mathbf{Z}, \geq)$ **poset**

11) Find the lexicographic ordering of these strings of lowercase English letters:

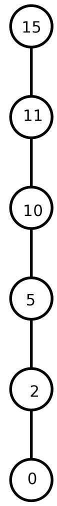
Solution:

(a) quack, quacking, quick, quicksand, quicksilver

(b) open, opened, opener, opera, operand,

(c) zero, zoo, zoological, zoology, zoom

12) Draw the Hasse diagram for the "less than or equal to" relation on $\{0, 2, 5, 10, 11, 15\}$.



13) Draw the Hasse diagram for inclusion on the set $\mathcal{P}(S)$ (the powerset of S) where $S = \{a, b, c, d\}$.