Binet's formula

The *Golden Mean* (or "Devine Proportion") has regarded since ancient times as the aesthetically ideal ratio of width to height for a rectangle. It is commonly reflected in Natural objects which grow by a linear increment (e.g. snail shells, sunflowers).

The golden mean is the limit,

$$\varphi = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} \approx 1.6180339887499\ldots$$

where f is the n^{th} member of the *fibonacci* sequence, 1, 1, 2, 3, 5, 8, 13, ..., credited to Leornardo of Pisa in the 12th Century.

The closed form for computing the n^{th} *Fibbonacci* number is attributed to Binet who published it in 1843. Others, including Knuth [Knuth Volume 1], say the original discovery was by de Moivre a century earlier. Define

$$f(n) = \begin{cases} 1 \text{ if } n = 1 \\ 1 \text{ if } n = 2 \\ f(n-1) + f(n-2), \text{ otherwise} \end{cases}$$

Proposition. For all $n > 0$,

$$f(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$