Foundations Qualifier (August 2004)

All languages are assumed to be over the alphabet $\{a, b, \$\}$.

1. For any language L, define the language $L' = \{xy \mid xay \in L\}$.
   Prove that if L is regular then so is L'.

2. Consider the language $L = \{a^{2^k} \mid k \geq 0\}$.

   (a) Prove or disprove that L is a regular language.

   (b) Prove or disprove that L is a context-free language.

3. Prove that the set generated by the following context-free grammar is $\{a, b\}^*$:

$$S \rightarrow bS \mid Sa \mid aSb \mid \epsilon$$

4. (a) The problem of deciding whether a recursively enumerable language is context-free is undecidable. Yet there is an effective syntax (the context-free grammars) that defines precisely the context-free languages. Discuss why these two statements are not in conflict.

   (b) Can the two statements above be made for the recursive languages, rather than the contex-free ones? Prove your answers.

5. Let f be a binary partial function over the natural numbers. Let g be the partial function defined by:
   (1) if $f(x, n) = 0$
   while for all $m < n$ the value $f(x, m)$ is defined and $\neq 0$,
   then $g(x) = n$.
   (2) if no n as above exists, then $g(x)$ is undefined.

   Prove that if f is computable, then so is g.

6. Argue whether the following languages of Turing machine codes are decidable or undecidable:

   (a) $\{M \mid M$ moves left at least once on every input $\}$.

   (b) $\{M \mid$ for all machines M', if $L(M') = L(M)$ then $|M'| \leq |M|\}$.

   (c) $\{M \mid L(M)$ is $NP\}$.

   (d) $\{M_1\$M_2 \mid L(M_1) \subseteq L(M_2)\}$.

7. The PARTITION problem is the following:

   > *Given a finite set A of positive integers whose sum is $2n$, can A be partitioned into two subsets so that the sum of the integers in each subset is equal to n.*

   The KNAPSACK problem is the following:

   > *Given a finite set V, two functions c and p from V into the natural numbers, and two natural numbers m and n, does a subset $U \subseteq V$ exist such that*

   $$\sum_{u \in U} p(u) \geq m \qquad \text{and} \qquad \sum_{u \in U} c(u) \leq n$$

   You are given as fact that the PARTITION problem is NP-complete. Prove that the KNAPSACK problem is also NP-complete.