PhD Qualifying Exam in Foundations

August 23, 2005

1. Show directly that if a language is defined by a regular expression then it is context-free.

2. Let M be a deterministic finite automaton over the alphabet $\{\mathtt{a}, \mathtt{b}\}$. Show that the following language is regular:

   $\{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid$ the run of M on w goes through each accepting state of M at least once $\}$

3. Consider the language L over the alphabet $\{\mathtt{a}, \mathtt{b}\}$ defined by

   $$L = \{\mathtt{a}^n\mathtt{b}\mathtt{a}^{n+1} \mid n \geq 0\}.$$

   (a) Give a context-free grammar that generates L. Prove that every string in L is generated by your grammar (you need not prove the converse implication).

   (b) Prove that L is not regular. For full credit use the Myhill-Nerode Theorem. For half credit use the Rabin-Scott Pumping Lemma.

4. For each of the following problems, state whether it is decidable and either provide a decision algorithm, or prove that none exists. You may quote major theorems without proof.

   (a) Does a given deterministic finite automaton accept all strings over its alphabet?

   (b) Is the language accepted by a given Turing machine regular?

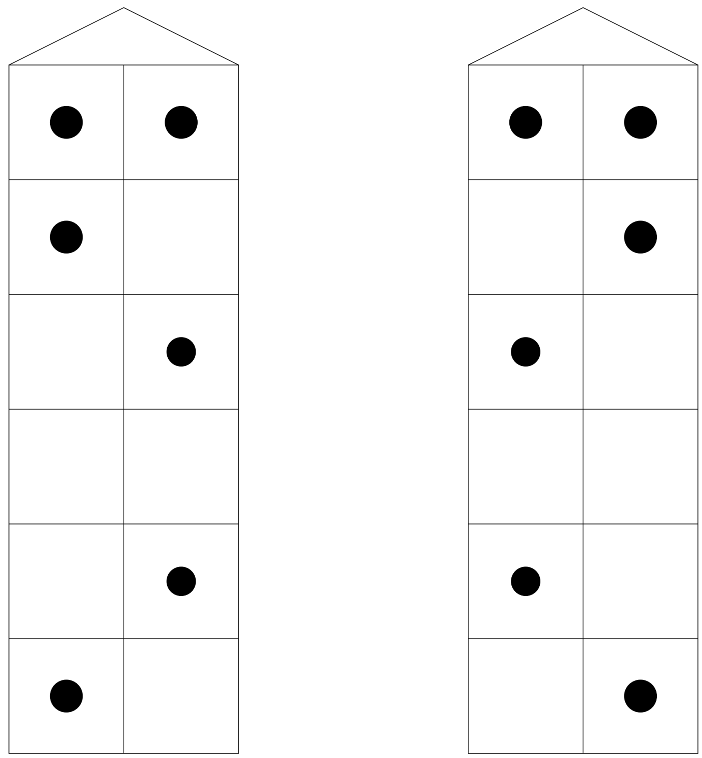
5. Let P be the collection of languages decidable in polynomial time. Under which of the following operations is P closed: (a) intersection; (b) complementation; (c) Kleene's star. Prove your answers succinctly.

6. Give an effective enumeration $M_1, M_2, \ldots$ of some Turing machines where the collection of these machines has the following two properties:

   (a) Each language $L(M_i)$ is NP.

   (b) Each NP language is $L(M_i)$ for some i.

7. Consider the following collection of puzzles. For $m \geq 1$ and $n \geq 1$, an m-n puzzle consists of a deck of m cards, each displaying a $2 \times n$ grid, with the "top" of the card identified by a triangular cap. Each square of the grid may have a circular hole punched at its center. Here is an example of a possible card in a m-n puzzle where $n = 6$:



A 6–puzzle card   Same card flipped

A deck of such cards may have a "see-through" hole, when a certain position of the grid has a hole on all the cards. A puzzle is *"solvable"* if it is possible to flip some of the cards horizontally (flip 180 degrees along the vertical axis, as shown in the figure) to yield a deck with no see-through holes in the *left* column.

Show that the collection of solvable m-n puzzles is NP-complete.