Visualizing Quaternions

Part III: Quaternion Frames

Andrew J. Hanson
Indiana University

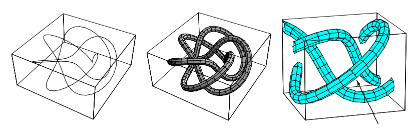
Part III: OUTLINE

- **Quaternion Curves:** generalize the Frenet Frame
- **Quaternion Frame Evolution**
- **Quaternion Curve and Surface Optimization**

What are Frames used For?

- Move objects and object parts in an animated scene.
- Move the camera generating the rendered viewpoint of the scene.
- Attach tubes and textures to thickened lines, oriented textures to surfaces.
- Compare shapes of similar curves.

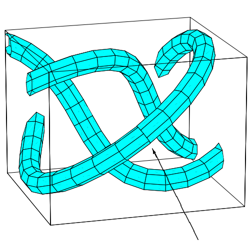
Motivating Problem: Framing Curves



The (3,5) torus knot.

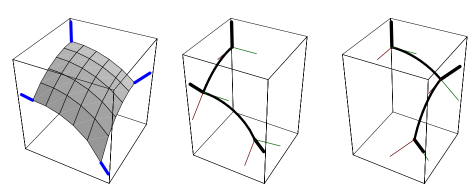
- Line drawing $\approx$ useless.
- Tubing based on parallel transport, **not periodic.**
- Closeup of the non-periodic mismatch.

Motivating Problems: Curves



Closeup of the non-periodic mismatch.
Can't apply texture.

Motivating Problems: Surfaces



A smooth 3D surface patch: two ways to get bottom frame.

No unique orthonormal frame is derivable from the parameterization.

3D Curves: Frenet and PT Frames

Now give more details of 3D frames: Classic Moving Frame:

$$\begin{bmatrix}\mathbf{T}'(t)\\ \mathbf{N}'(t)\\ \mathbf{B}'(t)\end{bmatrix} = \begin{bmatrix} 0 & k_1(t) & k_2(t)\\ -k_1(t) & 0 & \sigma(t)\\ -k_2(t) & -\sigma(t) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(t)\\ \mathbf{N}(t)\\ \mathbf{B}(t)\end{bmatrix} .$$

Serret-Frenet frame: $k_2 = 0$, $k_1 = \kappa(t)$ is the curvature, and $\sigma(t) = \tau(t)$ is the classical torsion. LOCAL.

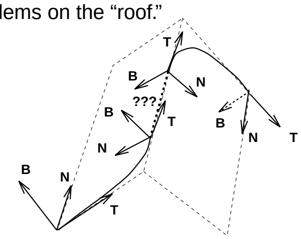
Parallel Transport frame (Bishop): $\sigma = 0$ to get minimal turning. NON-LOCAL = an INTEGRAL.

3D curve frames, contd

Frenet frame is *locally* defined, e.g., by

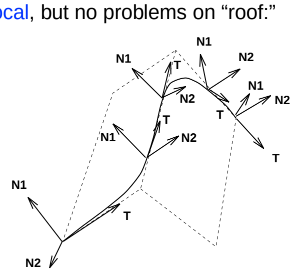
$$\mathrm{B}(t) = \frac{\mathbf{x}'(t) \times \mathbf{x}''(t)}{\|\mathbf{x}'(t) \times \mathbf{x}''(t)\|}$$

but has problems on the "roof."

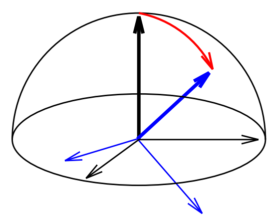


3D curve frames, contd

Bishop's Parallel Transport frame is *integrated over whole curve*, non-local, but no problems on "roof:"

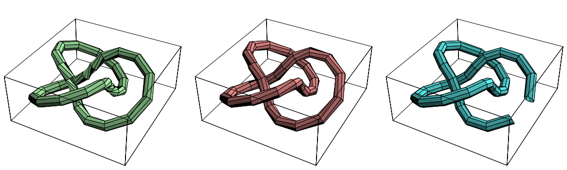


3D curve frames, contd



Geodesic Reference Frame is the frame found by tilting North Pole of "canonical frame" along a great circle until it points in desired direction (tangent for curves, normal for surfaces).

Sample Curve Tubings and their Frames



Tubings based on Frenet, Geodesic Reference, and Parallel Transport frames.
Easily see PT has least "Twist," but lacks periodicity.

3D Frames to Quaternion Frames

- **Unit four-vector.** Take $q = (q_0, q_1, q_2, q_3) = (q_0, \mathbf{q})$ to obey constraint $q \cdot q = 1$.
- **Multiplication rule.** Let $q * p$ be the quaternion product of two quaternions q and p, where

$$\begin{bmatrix}[q*p]_0\\ [q*p]_1\\ [q*p]_2\\ [q*p]_3\end{bmatrix} = \begin{bmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3\\ q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2\\ q_0p_2 + q_2p_0 + q_3p_1 - q_1p_3\\ q_0p_3 + q_3p_0 + q_1p_2 - q_2p_1\end{bmatrix}$$

. . . to Quaternion Frames . . .

- **Quaternion Correspondence.** The unit quaternions q and $-q$ correspond to a single 3D rotation $R_3(q)$:

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & q_0^2-q_1^2+q_2^2-q_3^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

- **Rotation Correspondence.** Let $q = (\cos\frac{\theta}{2}, \hat{\mathbf{n}}\sin\frac{\theta}{2})$, with $\hat{\mathbf{n}}$ a unit 3-vector, $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1$. Then $R(\theta, \hat{\mathbf{n}})$ is usual 3D rotation by θ in the plane perpendicular to $\hat{\mathbf{n}}$.

Quaternion Frame Evolution

Just as in 2D, let columns of $R_3(q)$ be a **frame**: $(\mathbf{T}, \mathbf{N}, \mathbf{B})$; this system has *nine components*.

Derivatives of the i-th column R_i in quaternion coordinates have the form:

$\dot{R}_i = 2W_i \cdot [\dot{q}(t)]$ where $i = 1, 2, 3$ and, e.g.,

$$W_1 = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix}$$

(rows form mutually orthonormal basis).

Quaternion Frame Evolution . . .

When we simplify by eliminating W_i . . .

we find the *square root* of the 3D frame eqns!

Tait (1890) derived the quaternion equation that makes **all 9 3D frame equations reduce to:** $\boxed{\dot{q} = (1/2)q * (0, \mathrm{k})}$ or:

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & k_2 & -k_1 & -\sigma \\ -k_2 & 0 & \sigma & -k_1 \\ k_1 & -\sigma & 0 & -k_2 \\ \sigma & k_1 & k_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Quaternion Frames . . .

Properties of Tait's quaternion frame equations:

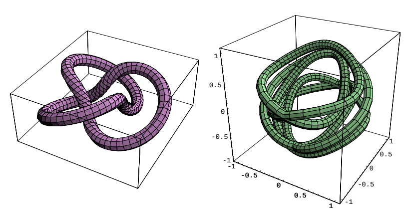
- Antisymmetry $\Rightarrow$ $q(t) \cdot \dot{q}(t) = 0$ as required to keep constant unit radius on 3-sphere.
- *Nine equations and six constraints* become *four equations and one constraint*, keeping quaternion on the 3-sphere. $\Rightarrow$ **Good for computer implementation.**
- MATHEMATICA code implementing this differential equation is provided.

Quaternion Frames . . .

- Analogous treatment (given in Hanson Tech Note in Course Notes) applies also to the Weingarten equations, allowing a direct quaternion treatment of the classical differential geometry of *surfaces* as well.

Example of a Quaternion Frame Curve

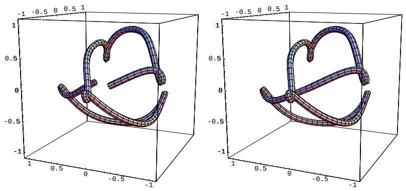
Left Curve = torus knot tubed with Frenet frame; Right Curve is projection from 4D of (twice around) quaternion Frenet frames:



see Notes: Hanson and Ma, "Quaternion Frame Approach to Streamline Visualization," *IEEE Trans. on Visualiz. and Comp. Graphics*, **1**, No. 2, pp. 164–174 (June, 1995).

Minimizing Quaternion Length Solves Periodic Tube

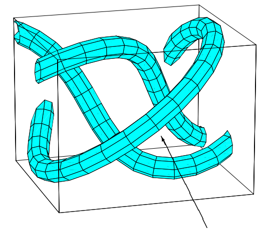
Quaternion space optimization of the non-periodic parallel transport frame of the (3,5) torus knot.



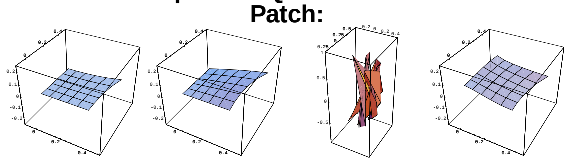
see Notes: “Constrained Optimal Framings of Curves and Surfaces using Quaternion Gauss Maps,” *Proceedings of Visualization ’98*, pp. 375–382 (1998).

Minimizing Quaternion Length Works

Result of Quaternion space optimization of the (3,5) torus knot frame.



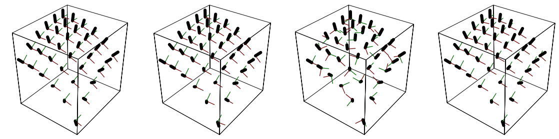
Can also Optimize Quaternion Frames on Patch:



(a) (b) (c) (d)

Quaternion frames for (a) Geodesic Ref. (b) One edge Parallel Transport. (c) Random. (d) Minimal area result.

3D Frames for Patch



(a) (b) (c) (d)

Quaternion frames for (a) Geodesic Ref. (b) One edge Parallel Transport. (c) Random. (d) Minimal area result.

SUMMARY

- **Quaternions can represent frames.**
- **Curve frames ⇒ quaternion curves.**
- **Surface patch frames ⇒ quaternion surface patches.**
- **Minimizing quaternion length or area finds parallel transport “minimal turning” set of frames.**