

# *Visualizing Quaternions*

## **Part III: Quaternion Frames**

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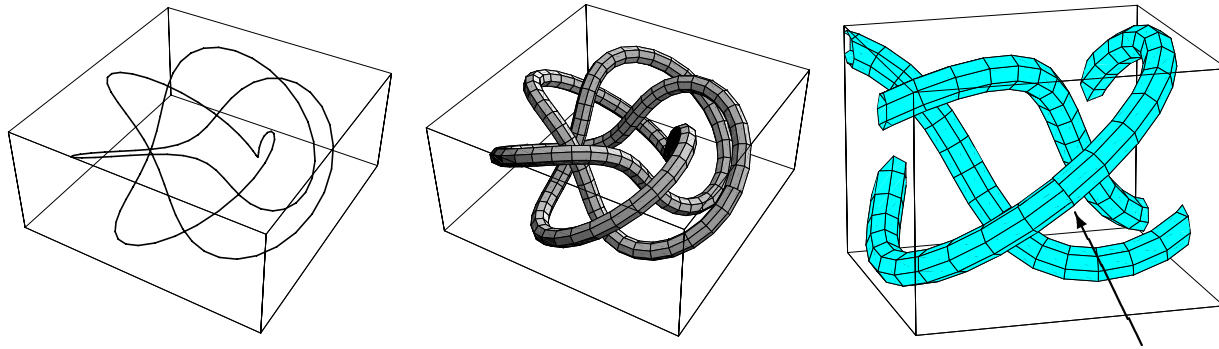
## Part III: OUTLINE

- **Quaternion Curves:** generalize the Frenet Frame
- **Quaternion Frame Evolution**
- **Quaternion Curve and Surface Optimization**

# What are Frames used For?

- Move objects and object parts in an animated scene.
- Move the camera generating the rendered viewpoint of the scene.
- Attach tubes and textures to thickened lines, oriented textures to surfaces.
- Compare shapes of similar curves.

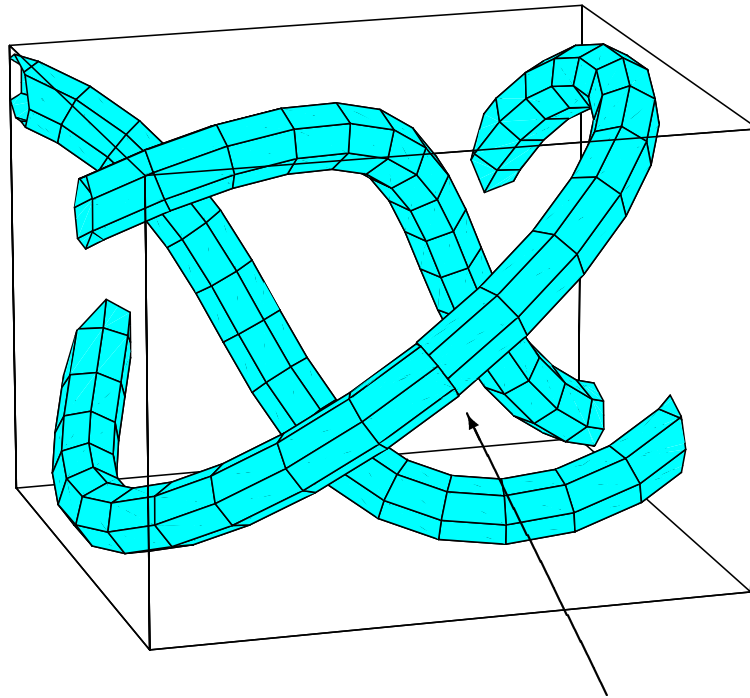
# Motivating Problem: Framing Curves



The (3,5) torus knot.

- Line drawing  $\approx$  useless.
- Tubing based on parallel transport, **not periodic.**
- Closeup of the non-periodic mismatch.

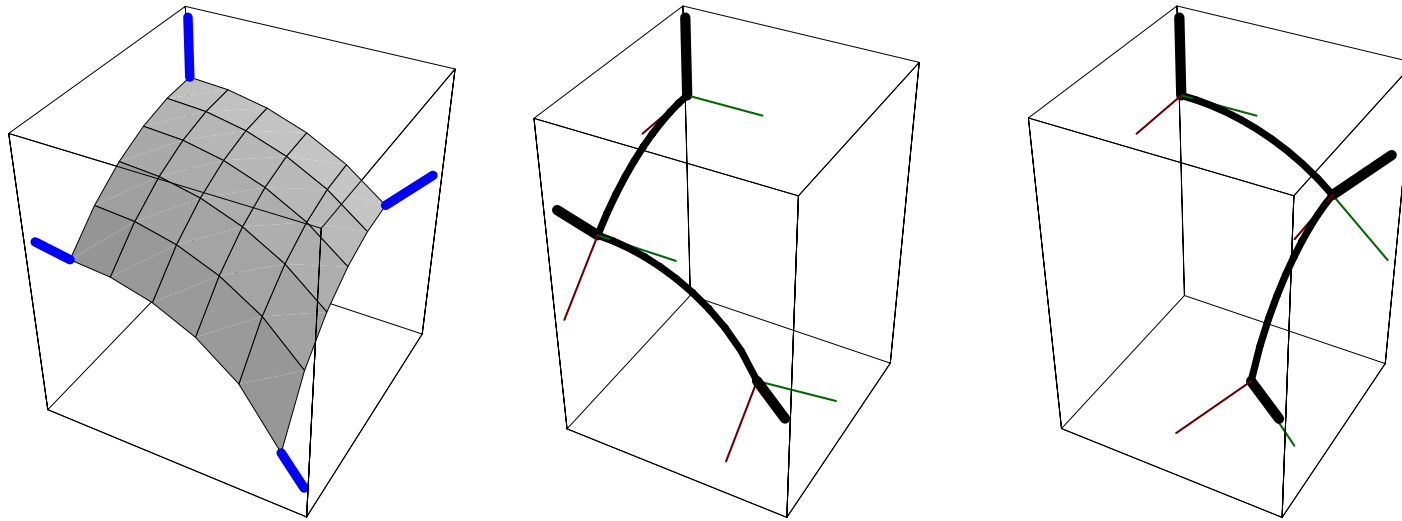
# *Motivating Problems: Curves*



Closeup of the non-periodic mismatch.

*Can't apply texture.*

# *Motivating Problems: Surfaces*



A smooth 3D surface patch: two ways to get bottom frame.

*No unique orthonormal frame is derivable from the parameterization.*

# 3D Curves: Frenet and PT Frames

Now give more details of 3D frames: Classic Moving Frame:

$$\begin{bmatrix} \mathbf{T}'(t) \\ \mathbf{N}'(t) \\ \mathbf{B}'(t) \end{bmatrix} = \begin{bmatrix} 0 & k_1(t) & k_2(t) \\ -k_1(t) & 0 & \sigma(t) \\ -k_2(t) & -\sigma(t) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(t) \\ \mathbf{N}(t) \\ \mathbf{B}(t) \end{bmatrix} .$$

*Serret-Frenet frame:*  $k_2 = 0$ ,  $k_1 = \kappa(t)$  is the curvature, and  $\sigma(t) = \tau(t)$  is the classical torsion. **LOCAL**.

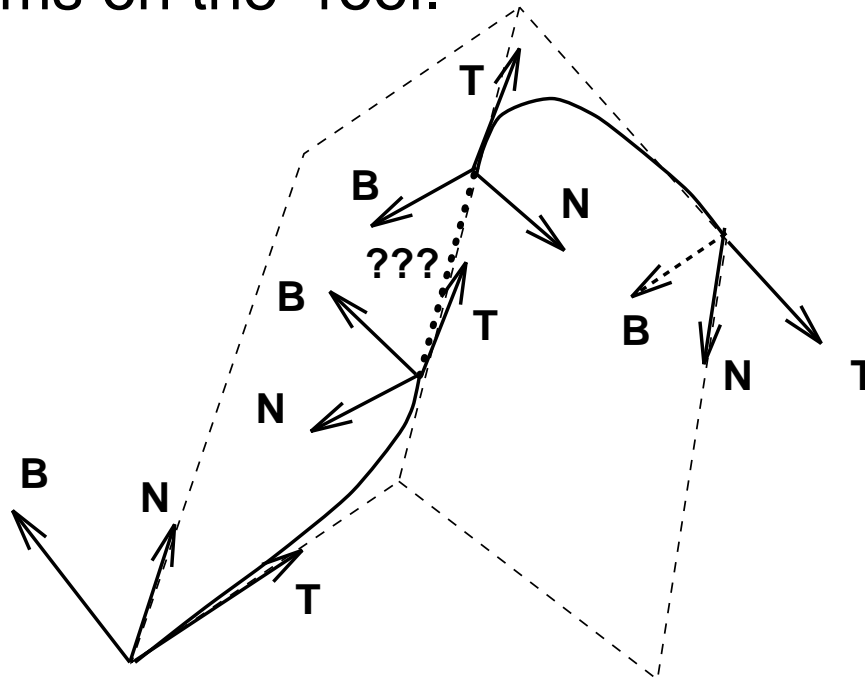
*Parallel Transport frame (Bishop):*  $\sigma = 0$  to get minimal turning. **NON-LOCAL = an INTEGRAL**.

## 3D curve frames, contd

Frenet frame is *locally* defined, e.g., by

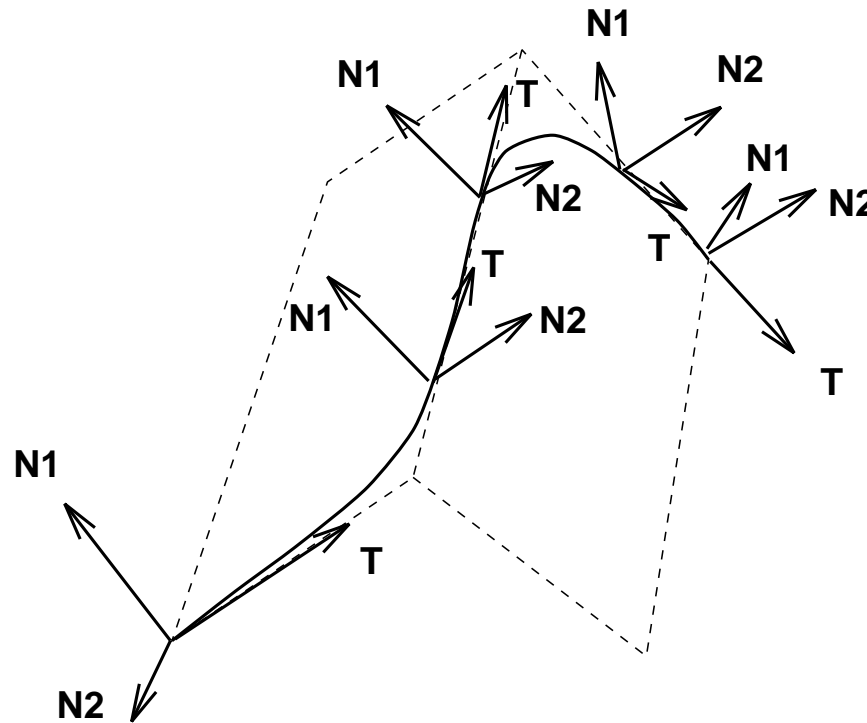
$$\mathbf{B}(t) = \frac{\mathbf{x}'(t) \times \mathbf{x}''(t)}{\|\mathbf{x}'(t) \times \mathbf{x}''(t)\|}$$

but has problems on the “roof.”

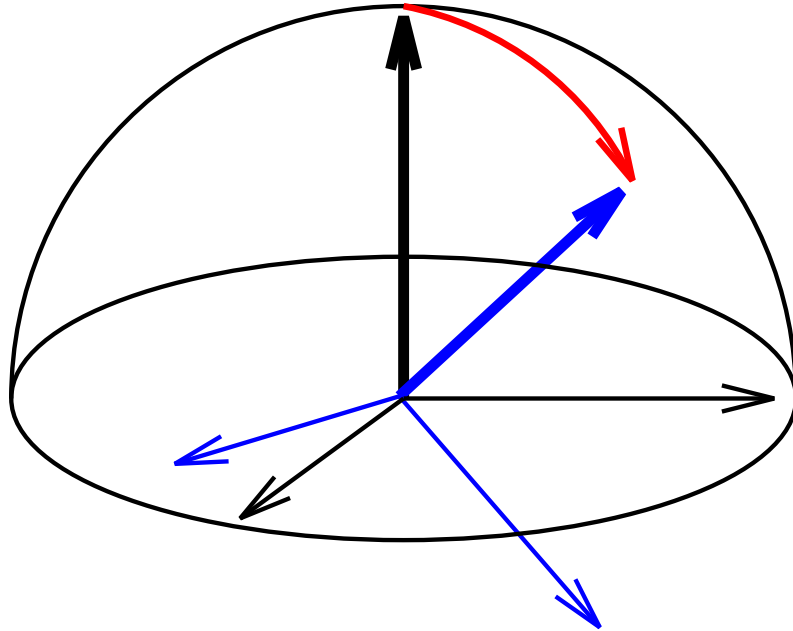


## 3D curve frames, contd

Bishop's **Parallel Transport frame** is *integrated over whole curve*, **non-local**, but no problems on “roof:”

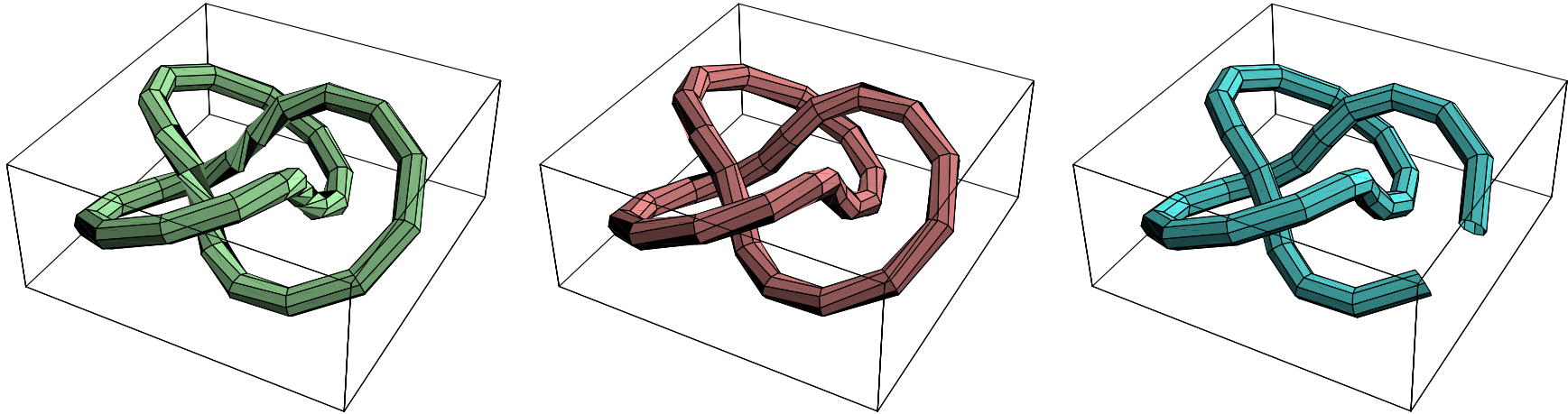


## *3D curve frames, contd*



**Geodesic Reference Frame** is the frame found by tilting North Pole of “canonical frame” along a great circle until it points in desired direction (**tangent for curves**, **normal for surfaces**).

## *Sample Curve Tubings and their Frames*



Tubings based on Frenet, Geodesic Reference, and Parallel Transport frames.

Easily see PT has least “Twist,” but lacks periodicity.

## 3D Frames to Quaternion Frames

- **Unit four-vector.** Take  $q = (q_0, q_1, q_2, q_3) = (q_0, \mathbf{q})$  to obey constraint  $q \cdot q = 1$ .
- **Multiplication rule.** Let  $q * p$  be the quaternion product of two quaternions  $q$  and  $p$ , where

$$\begin{bmatrix} [q * p]_0 \\ [q * p]_1 \\ [q * p]_2 \\ [q * p]_3 \end{bmatrix} = \begin{bmatrix} q_0 p_0 - q_1 p_1 - q_2 p_2 - q_3 p_3 \\ q_0 p_1 + q_1 p_0 + q_2 p_3 - q_3 p_2 \\ q_0 p_2 + q_2 p_0 + q_3 p_1 - q_1 p_3 \\ q_0 p_3 + q_3 p_0 + q_1 p_2 - q_2 p_1 \end{bmatrix}$$

## ... to Quaternion Frames ...

- **Quaternion Correspondence.** The unit quaternions  $q$  and  $-q$  correspond to a single 3D rotation  $R_3(q)$ :

$$\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

- **Rotation Correspondence.**

Let  $q = (\cos \frac{\theta}{2}, \hat{n} \sin \frac{\theta}{2})$ , with  $\hat{n}$  a unit 3-vector,  $\hat{n} \cdot \hat{n} = 1$ . Then  $R(\theta, \hat{n})$  is usual 3D rotation by  $\theta$  in the plane perpendicular to  $\hat{n}$ .

# Quaternion Frame Evolution

Just as in 2D, let columns of  $R_3(q)$  be a **frame**:  $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ ; this system has *nine components*.

Derivatives of the  $i$ -th column  $R_i$  in quaternion coordinates have the form:

$\dot{R}_i = 2W_i \cdot [\dot{q}(t)]$  where  $i = 1, 2, 3$  and, e.g.,

$$W_1 = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix}$$

(rows form mutually orthonormal basis).

## Quaternion Frame Evolution ...

When we simplify by eliminating  $W_i$  ...

we find the *square root* of the 3D frame eqns!

Tait (1890) derived the quaternion equation that makes **all 9**

**3D frame equations reduce to:**  $\dot{q} = (1/2)q * (0, k)$  or:

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & k_2 & -k_1 & -\sigma \\ -k_2 & 0 & \sigma & -k_1 \\ k_1 & -\sigma & 0 & -k_2 \\ \sigma & k_1 & k_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

## Quaternion Frames ...

Properties of Tait's quaternion frame equations:

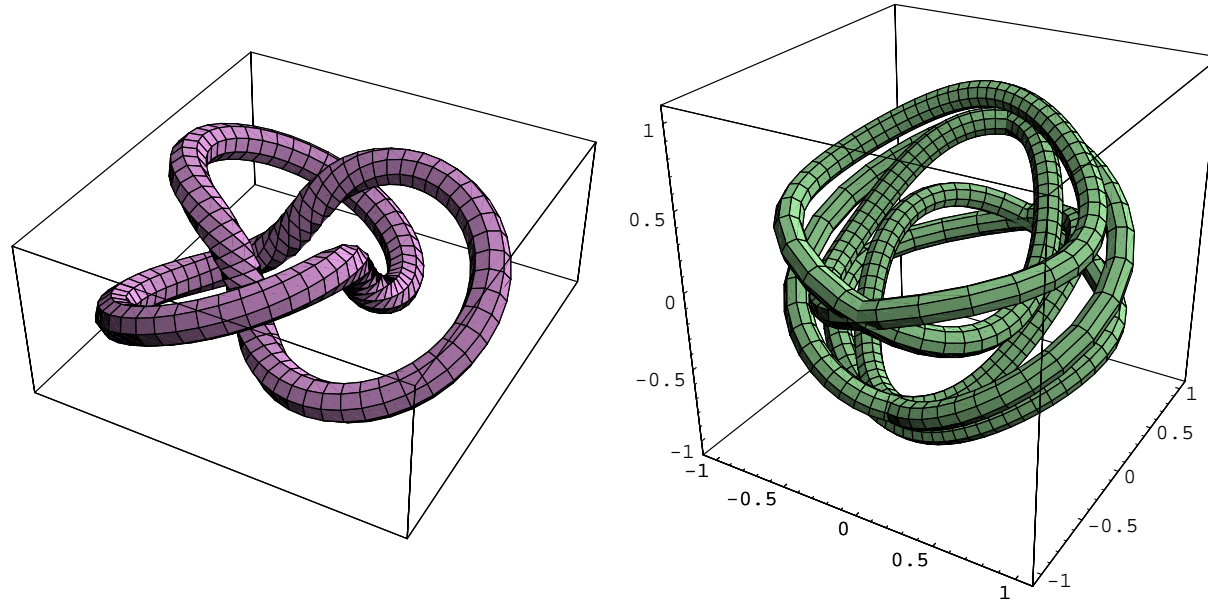
- Antisymmetry  $\Rightarrow q(t) \cdot \dot{q}(t) = 0$  as required to keep constant unit radius on 3-sphere.
- *Nine equations and six constraints* become *four equations and one constraint*, keeping quaternion on the 3-sphere.  $\Rightarrow$  **Good for computer implementation.**
- **MATHEMATICA** code implementing this differential equation is provided.

## *Quaternion Frames . . .*

- Analogous treatment (given in Hanson Tech Note in Course Notes) applies also to the Weingarten equations, allowing a direct quaternion treatment of the classical differential geometry of *surfaces* as well.

# Example of a Quaternion Frame Curve

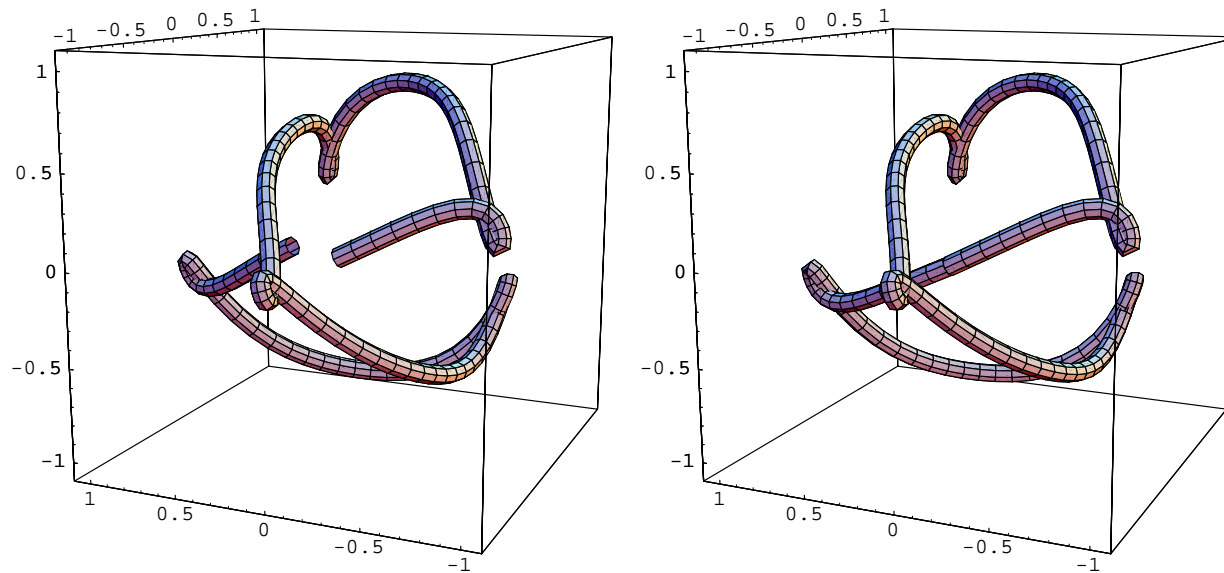
Left Curve = torus knot tubed with Frenet frame; Right Curve is projection from 4D of (twice around) quaternion Frenet frames:



*see Notes:* Hanson and Ma, “Quaternion Frame Approach to Streamline Visualization,” *IEEE Trans. on Visualiz. and Comp. Graphics*, **1**, No. 2, pp. 164–174 (June, 1995).

# Minimizing Quaternion Length Solves Periodic Tube

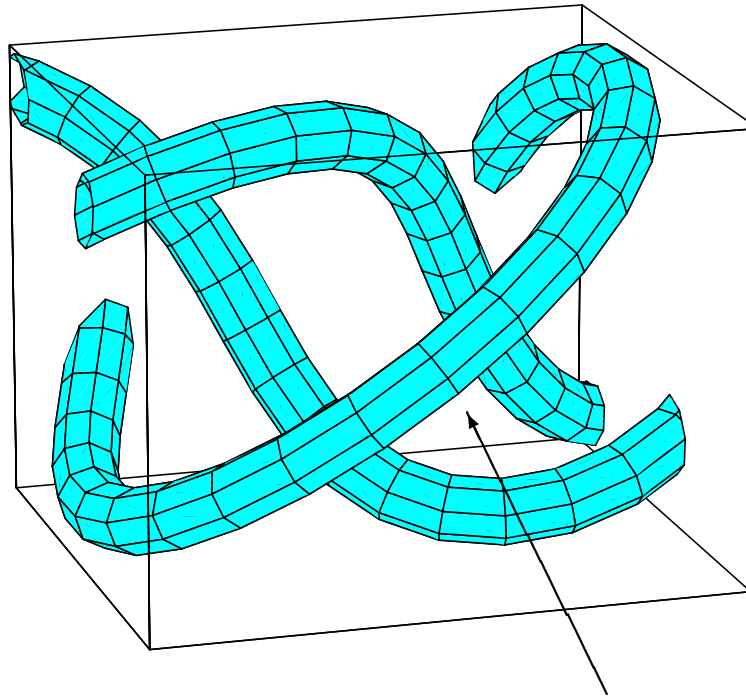
Quaternion space optimization of the non-periodic parallel transport frame of the (3,5) torus knot.



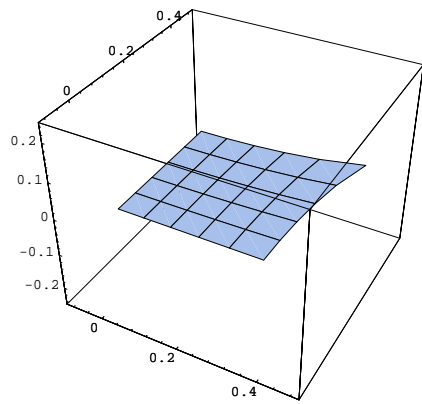
*see Notes:* “Constrained Optimal Framings of Curves and Surfaces using Quaternion Gauss Maps,” *Proceedings of Visualization '98*, pp. 375–382 (1998).

# Minimizing Quaternion Length Works

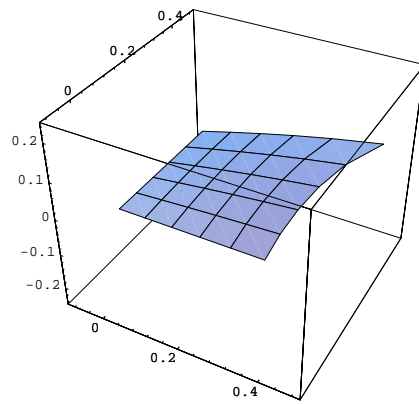
Result of Quaternion space optimization of the (3,5) torus knot frame.



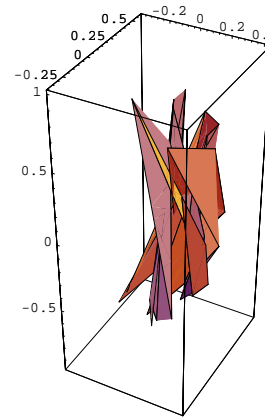
## Can also Optimize Quaternion Frames on Patch:



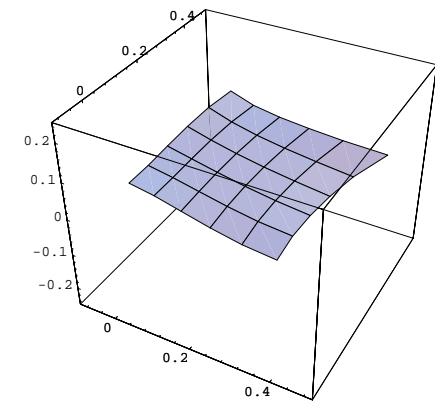
(a)



(b)



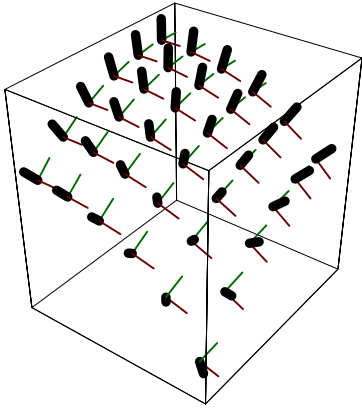
(c)



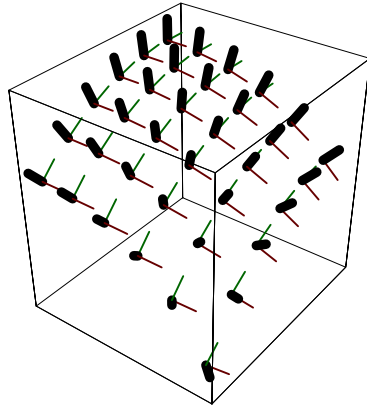
(d)

Quaternion frames for (a) Geodesic Ref. (b) One edge Parallel Transport. (c) Random. (d) Minimal area result.

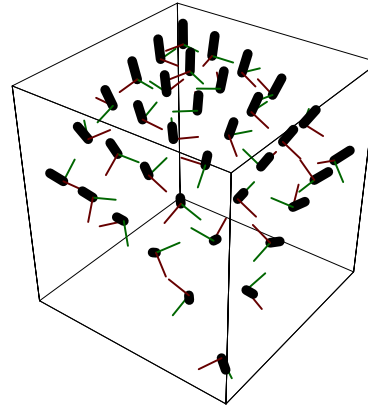
### 3D Frames for Patch



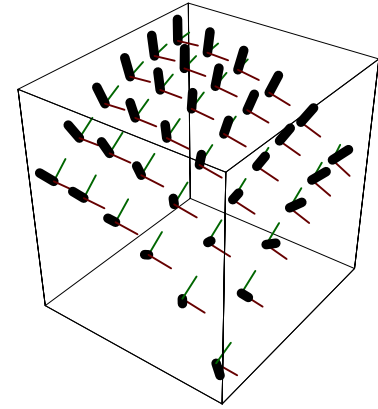
(a)



(b)



(c)



(d)

Quaternion frames for (a) Geodesic Ref. (b) One edge Parallel Transport. (c) Random. (d) Minimal area result.

# SUMMARY

- Quaternions can represent frames.
- **Curve frames  $\Rightarrow$  quaternion curves.**
- **Surface patch frames  $\Rightarrow$  quaternion surface patches.**
- **Minimizing quaternion length or area finds parallel transport “minimal turning” set of frames.**