Visualizing Quaternions

Part IV: Clifford Algebras

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Part IV: OUTLINE

- **Clifford Algebra:** Introduction to generalizing complex numbers and quaternions.
- **Reflections vs Rotations:** How to make a rotation.
- **Pin(N), Spin(N), O(N), and SO(N):** Double coverings of N-dimensional rotations.

Motivation

- **Quaternions are too special.** Complex numbers and quaternions run out of steam after dimensions 2,3,4.
- **Search for some generalizable idea.** How does the $(a^2 - b^2)$, etc., form generalize to N-dim rotation matrices?
- **Clifford Algebra:** Clifford found the generalization, but the *really* interesting relation to spin 1/2 elementary particles came much later.

Foundations

- N-dimensional space is described formally by basis vectors e_i, $i = 1, \ldots, N$ and real numbers v_i, so a vector looks like

$$V = \sum_i v_i e_i$$

- The *length* is computed from the inner product:

$$\|V\|^2 =< V, V >= \sum_{ij} v_i g_{ij} v_j$$

where g_{ij} is the identity matrix in Euclidean space.

- But the basis vectors obey a strange multiplication rule:

$$e_i e_j + e_j e_i = -2g_{ij}$$

- This is the CLIFFORD ALGEBRA.
- (Note: physicists would recognize these formulas as those obeyed by the *Pauli matrices* or the *Dirac matrices*.)

Clifford Algebra . . .

- How does this odd product concern us??

1. Because it contains *in any dimension* a way of expressing rotations as *multiple reflections about a plane*.

2. Because these expressions of rotations are natural *square roots* of the familiar $N \times N$ orthogonal matrix approach to writing rotations.

Clifford Algebra implements reflections

If $A = \sum a_i e_i$ is any vector with $\|A\| = 1$, then, using the Clifford multiplication rule,

$$V' = A * V * A = V - 2A < A, V >$$

This is just a *reflection* of the component of V lying in the direction of A about the *plane*

$$< A, X >= 0$$

Clifford Algebra – sample calculation

With $A = \sum a_i e_i$ and $\|A\| = 1$, compute V':

$$\begin{aligned}
A * V * A &= \sum_{i,j,k} a_i v_j a_k e_i e_j e_k \\
&= \sum_{i,j,k} a_i v_j a_k \left(-e_j e_i - 2g_{ij}\right) e_k \\
&= \sum_{i,j,k} a_i v_j a_k \left(-e_j \{e_i e_k\} - 2g_{ij} e_k\right)
\end{aligned}$$

$$= -V \sum_{i,j} a_i a_k \frac{1}{2}(e_i e_k + e_k e_i) - 2A < A, V >$$

Clifford Algebra – sample calculation, contd

Use *symmetrization trick* — only one-half of the symmetrized sum multiplying $\sum a_i a_k$ can survive; that in turn is just the Clifford rule right hand side!

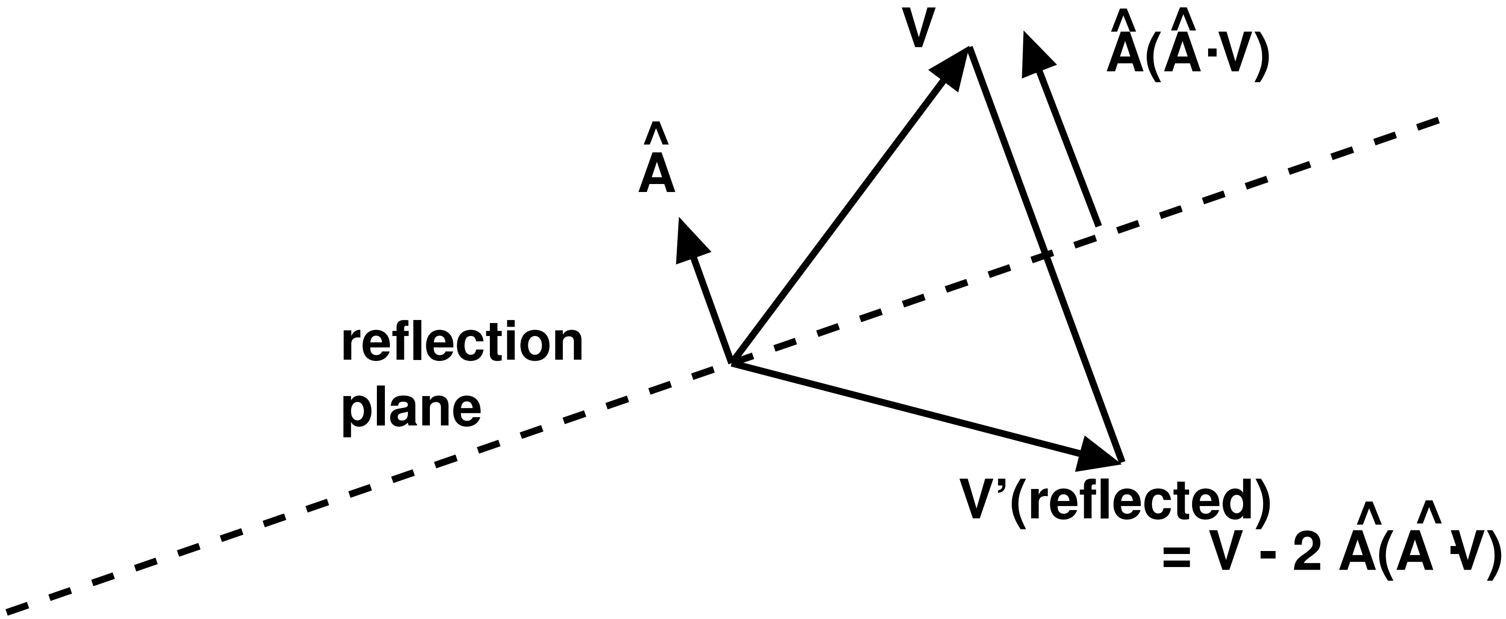
$$
\begin{aligned}
A * V * A &= -V \sum_{i,j} a_i a_k \frac{1}{2}(-2g_{ik}) - 2A < A, V > \\
&= +V < A, A > -2A < A, V >
\end{aligned}
$$

QED.

Note: The MINUS sign in $-2g_{ij}$ in the original definition makes the sign of V positive.

Clifford Algebra reflections

$$V' = A * V * A = V - 2A < A, V >$$



Clifford Algebra rotations

Let $B = \sum b_i e_i$ be *another* vector with $\|B\| = 1$:

Repeating the Clifford multiplication rule,

$$\begin{aligned} V'' &= B * V' * B \\ &= B * A * V * A * B \\ &= V' - 2B < B, V' > \\ &= V - 2A < A, V > \end{aligned}$$

$$-2B < B, V > +4B < B, A >< A, V >$$

Clifford Algebra rotations

This can be shown (e.g, in Mathematica) to be a *proper rotation* of the vector V, that is

$$V'' = B * A * V * A * B = \sum_{ij} R_{ij} v_j e_i$$

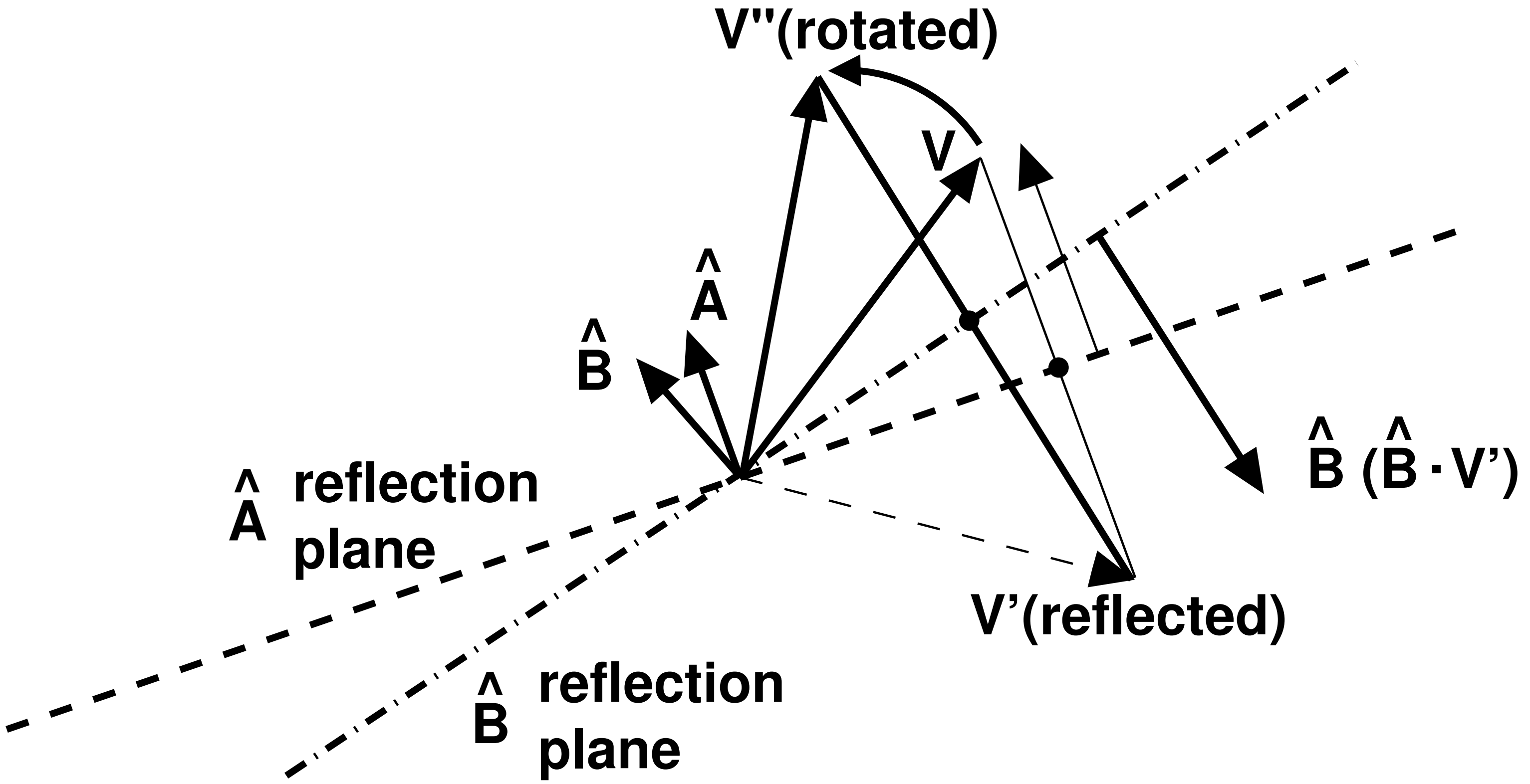
where R_{ij} is an orthonormal matrix of unit determinant.

Clifford Algebra rotations

Graphically, we have this:

$$\begin{aligned} V'' &= B * A * V * A * B \\ &= \sum_{ij} R_{ij} v_j e_i \end{aligned}$$

Clifford Algebra rotation graph



Clifford Algebra rotations

NOTE that in higher dimensions, you may need *more* than a single $(A,\, B)$ pair to exhaust all possible rotations:

N	pairs	params	constraints	freedom
2	1	$2*2$	3	1
3	1	$2*3$	3	3
4	2	$4*4$	10	6
5	2	$4*5$	10	10
6	3	$6*6$	21	15
7	3	$6*7$	21	21
.	.	.	.	.
N	$p = \lfloor \frac{N}{2} \rfloor$	$2p * N$	$p(2p+1)$	$\frac{N(N-1)}{2}$

Examples of Clifford Algebras

- $N = 1$: The basis $(1, e_1)$ with $(e_1)^2 = -1$ is just the **complex numbers.** But be careful: there is only one dimension, so the only possible reflection is $x \to -x$. This is *not* enough to do 2D rotations!

Examples of Clifford Algebras . . .

- $N = 2$**:** The basis $(1, e_1, e_2, e_1e_2)$ exhausts all possible Clifford products.
  Since $e_1e_2e_1e_2 = -1$, we can identify this basis with the **quaternions** $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$! But be careful: there are only *two* dimensions, so this is *not* enough to do 3D rotations!

- **Where is 2D Rot?** *True* basis of rotations is the **even part** of the family of all Clifford products, or $\boxed{(1, e_1e_2)}$!!

2D Rotations done right

- **What is i?** What we *called* $i = \sqrt{-1}$ is *really* $\boxed{i = e_1 e_2}$

- **How do we rotate in 2D?** Let $R = a + b e_1 e_2, \qquad R^\dagger = a - b e_1 e_2$:

$$\boxed{R * V * R^\dagger = V' = v'_1 e_1 + v'_2 e_2}$$

where V' now means a rotation, and

$$\begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = \begin{bmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

2D Rotations done right . . .

So the half-angle formula is mandatory! Our 2D transformation was not so silly after all; nothing else generalizes to N-dimensions. The Clifford algebra for $N = 2$ automatically produces:

$$R_2(a, b) = \begin{bmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{bmatrix} ,$$

where $a^2 + b^2 = 1$, and we have the solution
$a = \cos(\theta/2),\ b = \sin(\theta/2)$.

3D Rotations done right

3D is of course a little trickier: here the full basis of all Clifford products is 8-dimensional:

$$\boxed{(1, e_1, e_2, e_3, e_2e_3, e_3e_1, e_1e_2, e_1e_2e_3)}$$

- **Even part is rotations.** To exclude reflections, we keep only the even part:

$$\boxed{(1, e_2e_3, e_3e_1, e_1e_2)}$$

- **These are the quaternions:** identify these with $(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$.

- **General 3D Rotation:** with

$$R = q_0 + q_1 e_2 e_3 + q_2 e_3 e_1 + q_3 e_1 e_2$$

we have

$$R * V * R^{\dagger} = \sum_{i=1}^{4} v_i' e_i$$

where the coefficients of v_i in v_i' are *precisely* our old quaternion formula.

Higher dimensions:

As one might expect, higher dimensions are *much more complicated,* and do not work out so neatly, except for a convenient accident in $N = 4$, which allows a "double-quaternion" form (see "Rotations for N-dimensional Graphics").

Higher dimensions:

But we can do a little counting to see what is going on in N dimensions, where we know that the number of rotational degrees of freedom is

$$\boxed{N(N-1)/2}$$

Degrees of freedom in higher dimensional Spin representations:

N	Dim(even Clifford)	Dim(Rotations)	Constraints (the difference)
1	1	0	1
2	2	1	1
3	4	3	1
4	8	6	2
5	16	10	6
6	32	15	17
7	64	21	43
8	128	28	100
.	.	.	.
N	2^{N-1}	$N(N-1)/2$	$\frac{2^N - N^2 + N}{2}$

Pin(N), Spin(N), O(N), SO(N) and all that . . .

Spin representations of orthogonal groups follow from the Clifford Algebra Cl(N): (So do **spinors** — but another time...)

- **Pin(N).** G is "Pin" if it's a general reflection; G includes **all** elements of Cl(N).
- **Spin(N).** G is "Spin" if it's a general rotation; G is only **even** elements of Cl(N).
- **O(N).** $G * V * G^\dagger$ is "O" if G is in Pin and result is a vector reflection.
- **SO(N).** $G * V * G^\dagger$ is "SO" if G is in Spin and result is a vector rotation.

GRAND CONCLUSION

- **Rotation Matrices:** can be represented by a "square root" object with simpler geometric properties than rotations (= quaternions for $N = 2, 3, 4$).
- **Visualization of Quaternions:** is possible using sphere projection trick and a solid unit sphere.
- **Quaternion Curves, Surfaces, Volumes:** embedded in that sphere represent animations, flows, curve tubings, joint ranges, etc.
- **Clifford Algebras:** form the rigorous basis for the whole set of concepts.