A Illustration of the Insufficiency of Classic Methods of Classifying Tiling Spaces with Applications in the Theory of Quasi-Crystals

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Abstract

We describe substitution tilings and compare two topological invariants for tiling spaces, orbit equivalence and flow equivalence. Prefix problems are used as a source of examples where the two invariants differ in their description of the topology. A condition is given to establish the non-orbit equivalence of certain prefix problems and their solutions. We present potential applications of flow equivalence to the theory of quasi-crystals.

1. TILINGS

1.1. **Definitions.** Consider a one-dimensional infinite line. Suppose $\mathcal{S}$ is a set of line segments of different lengths, or tiles, denoted by $s_1 \dots s_N$.

Definition: A one-dimensional tiling $\mathcal{T}_{\mathcal{S}}$ is a bi-infinite sequence $\dots t_{-n} \dots t_0 \dots t_n \dots$, where the t_i are tiles in $\mathcal{S}$. These tiles cover the one-dimensional line.

2. SUBSTITUTION TILINGS

2.1. **Definitions.** Here we briefly describe substitution tilings; for a more thorough treatment see [1].

Let $\mathcal{A} = \{1, 2, \dots, \text{card}(\mathcal{A})\}$ and $\mathcal{B} = \{1, 2, \dots, \text{card}(\mathcal{B})\}$ be finite alphabets. $\mathcal{A}^*$ will denote the collection of finite nonempty words with letters in $\mathcal{A}$.

Definition: Given a map $\tau : \mathcal{A} \to \mathcal{B}^*$ define the associated substitution matrix $A_\tau = (a_{ij})_{i \in \mathcal{B}, j \in \mathcal{A}}$, where a_{ij} is the number of occurences of i in the word $\tau(j)$.

Definition: A substitution is a map $\varphi : \mathcal{A} \to \mathcal{A}^*$ with an element $a \in \mathcal{A}$ such that the first letter of $\varphi(a)$ is a. φ is primitive if $\varphi^n(i)$ contains j for all $i, j \in \mathcal{A}$ and sufficiently large n.

An important property is that the substitution matrix A_φ associated with a primitive substitution φ has an eigenvalue λ_φ larger in modulus than its other eigenvalues. λ_φ is called the Perron-Frobenius eigenvalue of A_φ [1, 3]. Later on we will denote the Perron-Frobenius eigenvalue by λ_u because it corresponds to the most unstable eigenvector of the matrix. In the remainder of this paper all substitutions are assumed to be primitive.

Definition: A substitution φ is proper if there exist $a, b \in \mathcal{A}$ such that for all $c \in \mathcal{A}$, $\varphi^n(c)$ begins with a and ends with b, for sufficiently large n.

Definition: A fixed word $\mathcal{W}$ for a substitution φ is a bi-infinite word such that $\varphi(\mathcal{W}) = \mathcal{W}$. If φ is proper it has a unique associated fixed word [3].

Fixed words of proper substitutions have many interesting properties too numerous to list here. Of particular interest, though, is that a fixed word $\mathcal{W}$ is uniformly recurrent. That is, given a sequence u in $\mathcal{W}$, the greatest difference of two successive occurences of u is bounded [3]. In addition, u occurs an infinite number of times in $\mathcal{W}$ [7].

Definition: Consider a fixed bi-infinite word $\mathcal{W}_\varphi$. An associated substitution tiling $\mathcal{T}_{\mathcal{W},\mathcal{S}}$ is defined by associating each element $a_i \in \mathcal{A}$ with a tile $s_i \in \mathcal{S}$.

3. INVARIANTS

Here we define two important invariants for topologies associated with tiling spaces. The first, orbit equivalence, is a group-theroretical approach based on properties of substitution matrices. The second, flow equivalence, is a combinatoric approach based on the substitutions themselves.

3.1. **Orbit equivalence.** Given a substitution φ and associated matrix M we can define a dimension triple $\mathcal{D}_\varphi$:

$$\mathcal{D}_\varphi = (G, G^+, \delta) \tag{1}$$

where G and G^+ are groups defined by

$$G = \{(a,b) \in \mathbb{Q} \times \mathbb{Q} : \left(M^n \begin{pmatrix} a \\ b \end{pmatrix}\right)^T \in \mathbb{Z} \times \mathbb{Z} \quad \text{for some} \quad n \geq 1\} \tag{2}$$

$$G^+ = \{(a,b) \in G : \left(M^n \begin{pmatrix} a \\ b \end{pmatrix}\right)^T \in \mathbb{Z}^+ \times \mathbb{Z}^+ \quad \text{for some} \quad n \geq 1\} \tag{3}$$

and δ is called the order unit. If φ is proper, $\delta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Definition: Associated with each substitution ς there is a Cantor set $\mathcal{X}_\varsigma$ and a "shift homeomorphism" σ_ς. Substitutions ς and φ are orbit equivalent if there is a homeomorphism $h : \mathcal{X}_\varsigma \to \mathcal{X}_\varphi$ with the property that $h(\{\sigma_\varsigma^n(x) : n \in\}) = \{\sigma_\varphi^n(h(x)) : n \in \mathbb{Z}\}$ for all $x \in \mathcal{X}_\varsigma$. It is a theorem of [4] that ς and φ are orbit equivalent iff the dimension triples associated with ς and φ are isomorphic with isomorphism ϕ:

$$\phi : \mathcal{D}_\varsigma \to \mathcal{D}_\varphi. \tag{4}$$

3.2. **Flow equivalence. Definition:** Two substitutions ς and φ are flow equivalent if the suspensions of σ_ς and σ_φ are homeomorphic. This is equivalent to the tiling spaces associated with ς and φ being homeomorphic. A complete combinatorial invariant for flow equivalence of substitutions is given in [1].

What we use here is the fact that if ς and φ are related by a process called rewriting, of which "solving a prefix problem", as described below, is a special case, then ς and φ are flow equivalent. "Solving a prefix problem" of ς results in substitutions φ and σ such that

$$\sigma \circ \varsigma = \varphi \circ \sigma \tag{5}$$

3.3. **Examples.** Consider the substitutions φ and ς defined by $\varphi(a) = ba, \varphi(b) = baa, \varsigma(a) = aba$, and $\varsigma(b) = a$. φ and ς are flow equivalent, since 5 holds when we define σ by $\sigma(a) = a, \sigma(b) = ab$.

4. PREFIX PROBLEMS

Since we wish to compare the two invariants given above, we need a source of pairs of substitutions to compare with each other. Substitutions with prefix problems can be rewritten in such a way that the fundamental structure of the substitution is preserved while simplifying the substitution. Prefix problems and their solutions provide the desired pairs of substitutions for our comparison below.

4.1. **Definition.** A substitution φ has a prefix problem if for some $i \neq j$, $\varphi(j)$ is a prefix of $\varphi(i)$ (that is, for some $i \neq j$, $\varphi(i) = \varphi(j)\mathcal{W}$ for some word $\mathcal{W}$).

4.2. **Examples.** Consider the substitution defined by $\varphi(a) = ab$ and $\varphi(b) = abb$. Then φ has a prefix problem since ab is a prefix of abb.

5. INVARIANTS COMPARED

5.1. **Conditions for orbit equivalence.** Any prefix problem can be rewritten in such a way that the rewriting and the original substitution are flow equivalent [1]. Here we find conditions for which these pairs are not orbit equivalent. It is our opinion that when these conditions hold, flow equivalence better captures the structure of the substitution pairs.

In this section we let A and B represent the substitution matrices for proper substitutions α and β with a prefix problem and its rewritten form, respectively. Denote the dimension triples for α and β by $(G_\alpha, G_\alpha^+, \delta_\alpha)$ and $(G_\beta, G_\beta^+, \delta_\beta)$. λ_u denotes the Perron-Frobenius eigenvalue for A, and v_u denotes its associated eigenvector. Let the isomorphism ϕ from 4 be represented by the linear transform S, given by

$$S = \begin{pmatrix} q & r \\ s & t \end{pmatrix} \tag{6}$$

In the next subsections we find conditions for which S is an isomorphism.

5.1.1. *Order unit condition.* Since S must be a dimension triple isomorphism, it needs to preserve the order unit, δ, and because α and β are proper substitutions, $\delta_\alpha = \delta_\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus we have

$$\begin{pmatrix} q & r \\ s & t \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7}$$

so we find conditions on r and t,

$$r = 1 - q \tag{8}$$

$$t = 1 - s \tag{9}$$

5.1.2. *Prefix problem condition.* Although there may be many ways to rewrite substitution α, we choose β such that it is a rewriting of α with associated substitution σ where $\sigma(a) = a$, $\sigma(b) = ba$. Now β is uniquely determined in terms of A. It can be verified that

$$\text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then} \quad B = \begin{pmatrix} a+b & b \\ c+d-a-b & d-b \end{pmatrix} \tag{10}$$

Note that this rewriting has preserved eigenvalues, and that $\det A = \det B$. Additionally, without loss of generality we only consider matrices A where $c + d > a + b$, because we can transform any substitution φ' that does not satisfy this condition into the correct form φ by applying the substitution μ defined by $\mu(a) = b, \mu(b) = a$. Finally, note that all elements of A are positive.

5.1.3. *Unstable eigenvector condition.* Because S must map the positive cone for A to the positive cone for B (see eqn 3), it must map $v_{u,A}$ to $v_{u,B}$. Thus we have the condition

$$S.v_{u,A} = \alpha\, v_{u,B} \qquad \text{where } \alpha \text{ is a real scalar} \tag{11}$$

Also, since A and B are $2x2$ matrices with all elements ≥ 0, we can write formulas for $v_{u,A}$ and $v_{u,B}$:

$$v_{u,A} = \begin{pmatrix} \frac{a-d-\sqrt{(a-d)^2+4bc}}{2c} \\ 1 \end{pmatrix} \tag{12}$$

$$v_{u,B} = \begin{pmatrix} \frac{a+2b-d-\sqrt{(a-d)^2+4bc}}{2(c+d-a-b)} \\ 1 \end{pmatrix} \tag{13}$$

Note how these eigenvector formulas explicitly place more restrictions on S.

5.1.4. *Determinant condition.* It is shown in [1] that a certain diagram commutes when two substitutions are flow equivalent, which is the case here. In particular, we have

$$S_1 = S\, A^m \tag{14}$$

$$T_1 = T\, B^n \tag{15}$$

where $S_1, T_1 \in \mathbb{Z} \times \mathbb{Z}$, $m, n \in \mathbb{Z}$, and $T = S^{-1}$. Thus $S, T \in \mathbb{Q} \times \mathbb{Q}$. Additionally,

$$\det S_1 = \det S\, (\det A)^m \tag{16}$$

$$\det T_1 = \det T\, (\det B)^n \tag{17}$$

Represent $\det S$ by p/q, where $(p, q) = 1$. Now write p as a prime factorization, $p = \prod p_i^{e_i}$, where the p_i are the prime factors of p and the e_i are their powers. Likewise, write $q = \prod q_i^{f_i}$. Since $S_1 \in \mathbb{Z} \times \mathbb{Z}$, $q_i | (\det A)$, where $x|y$ denotes divisibility of y by x. Similarly, since $\det T = (\det S)^{-}1$, we have $q_i | (\det A)$. But because B is the rewriting of A, $\det A = \det B$, so all the prime factors of p and q must divide $\det A$.

5.1.5. *Irrational eigenvector condition.* Unfortunately, the previous conditions are not enough to determine in general if S exists. However, by restricting ourselves to matrices A with irrational eigenvectors, enough conditions are available to show that S cannot be an isomorphism in many cases. We get this additional condition from 11 by equating the rational parts of the equation and by equating the irrational parts of the equation.

5.1.6. *Solution.* Solving the system determined by the previous conditions gives

$$S = \begin{pmatrix} \frac{c}{c+d-a-b} & \frac{a+b-d}{a+b-c-d} \\ 0 & 1 \end{pmatrix} \tag{18}$$

Now $\det S = \frac{c}{c+d-a-b}$, and by the determinant condition above, we have our main condition for matrix A:

$$c|(ad) \quad \text{and} \quad (c+d-a-b)|(ad-bc) \tag{19}$$

5.1.7. *Summary.* A dimension triple isomorphism S must necessarily satisfy the conditions listed above, so for S to have a chance to exist, A must satisfy 19. However, these conditions are not sufficient to show that S is an isomorphism. Additionally, remember that we have restricted ourselves to matrices A with irrational eigenvectors. Subject to this constraints, we can use the required form of S given above. If S does not fit this form, then we have a case where α and β correspond to flow equivalent substitutions that are not orbit equivalent, as desired.

5.2. **Examples.**

5.2.1. *Primary Example.* Consider the example in section 3.3. We have the associated substitution matrices

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \tag{20}$$

Here $c = 2$, but $\det A = -1$, so no isomorphism S exists, and A and B are not orbit equivalent. However, they are flow equivalent.

5.3. **Computer results.**

Condition 19 was tested on thousands of examples of substitution pairs subject to the constraints above. More than half of the examples tested were shown to be flow equivalent but not orbit equivalent, so it seems that flow equivalence does a good job at capturing structure in this sort of situation.

6. QUASI-CRYSTALS

Quasicrystals are noncrystalline solids first discovered in 1984 [8]. Quasicrystals have some structures that repeat, but no structure in the quasicrystal is perfectly periodic. In this regard, quasicrystals greatly resemble substitution tilings. Although the one-dimensional tilings studied here are of limited applicability to the two- or three-dimensional world of quasicrystals, this work may eventually help us understand the structure of quasicrystals better. The recurrent properties of substitution maps are not too dissimilar to the recursive processes which form quasicrystals in nature.

7. FUTURE DIRECTIONS

In general, comparing flow equivalence to orbit equivalence is tedious because the dimension group is hard to compute. Sometimes it can be written as a direct sum of extended integer rings, but this does not work in general. A straightforward algorithm for computing a dimension group would be helpful in this and related research. Note that this problem prevented us from finding a condition for orbit equivalence when A had rational eigenvectors, and it also prevented us from finding a sufficient condition for the existence of an isomorphism S. Both of these conditions should be sought in future work. In addition, there are more results along the lines of this paper if A is unimodular. In this case, a necessary and sufficient condition on A can probably be calculated in general, for both rational and irrational eigenvectors. Finally, these results should be extended to higher dimensions, although significantly different techniques need to be used in the general case because we relied on explicit computation of eigenvectors. Higher dimensional results would have a better chance of yielding useful results in the study of quasicrystals. In general, much work remains to be done on equivalences of nonunimodular matrices and on matrix equivalence in higher dimensions.

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