For full credit, each answer should be in the simplest form possible. All random selections are over a uniform distribution.

1. Give the value of each of the following expressions.
   1a. $503 \mod 3$.
   1b. $4!$.
   1c. $\binom{10}{2}$.
   1d. $\phi(100)$, where $\phi$ is the Euler phi function.

2. Define the length of an integer to be the number of digits needed to write it in binary without using leading zeros. Thus, the integer 0 has length zero, the integer 1 has length one, the integer 2 has length two, the integer 3 has length two, etc. Suppose you randomly select an $n$ bit integer (range 0 to $2^n - 1$).

   2a. What is the probability that the selected integer has length $k$?

   2b. What is the average length of the selected integer?

3. You play a game where you roll a six sided die. You always roll the die at least one time. Each time, if the result of the roll is a one, then you roll the die again.

   3a. What is the probability that you roll the die $i$ times?

   3b. What is the average number of times you roll the die?

   3c. Your score in the game is the sum of the numbers from all of your rolls. What is your average score?

   3d. What is the variance of your score?

4. An algorithm builds a doubly linked list. The list is initially empty. The algorithm inserts $k$ distinct integers onto the list, where each integer is in the range 1 to $n$. The algorithm remembers the value of the first integer that it inserts onto the list. After the first integer, the algorithm inserts those integers less than the first integer on the left side of the list. It inserts those integers greater than the first integer on the right side of the list.

   4a. Let $p_{ijkn}$ be the probability that the first integer is $j$ and that $i$ of the $k$ integers are inserted on the left side of the list. (The first integer is not counted as being inserted on either side.) Give a formula for $p_{ijkn}$.

   4b. What is the average number of integers inserted on the right side of the list?