Selected Solutions

(7) Let $X$ be a subset of $R$ and let $r$ and $s$ be relations on $R$ (i.e., $r$ is the relation $r(R)$ and $s$ is the relation $s(R)$). Prove or disprove the following equalities.

(a) $\pi_X(r \cap s) = \pi_X(r) \cap \pi_X(s)$
(b) $\pi_X(r \cup s) = \pi_X(r) \cup \pi_X(s)$
(c) $\pi_X(r - s) = \pi_X(r) - \pi_X(s)$

(a) False. Counterexample: $r = \{(x, y)\}, s = \{(x, z)\}$ and $R = \{A, B\}, X = \{A\}$. Then $\pi_X(r \cap s) = \emptyset$ but $\pi_X(r) \cap \pi_X(s) = \{(x)\}$.

(b) True. $\pi_X(r) = \{t \mid \exists v \in r \text{ such that } \pi_X(\{v\}) = \{t\}\}$ and $\pi_X(s) = \{t \mid \exists v \in s \text{ such that } \pi_X(\{v\}) = \{t\}\}$. Hence, $\pi_X(r) \cup \pi_X(s) = \{t \mid (\exists v \in r \text{ such that } \pi_X(\{v\}) = \{t\}) \text{ or } (\exists v \in r \text{ such that } \pi_X(\{v\}) = \{t\})\} = \{t \mid \exists v \in r \cup s \text{ such that } \pi_X(\{v\}) = \{t\}\} = \pi_X(r \cup s)$. Note that the “or” and “and” connectives are indeed those from propositional logic and that one could write $\lor$ and $\land$ instead.

(c) False. Use counterexample from (a).

(8) Let $r$ and $r'$ be relations on $R$, and let $s$ be a relation on $S$. Prove or disprove:

(a) $(r \cap r') \bowtie s = (r \bowtie s) \cap (r' \bowtie s)$
(b) $(r - r') \bowtie s = (r \bowtie s) - (r' \bowtie s)$
(a) The statement is correct. Prove: \((r \cap r') \bowtie s = \pi_{R \cup S}(\{t \mid t \in (r \cap r') \times s \text{ and } \forall A, B \in R \cap S : t[A] = t[B]\}) = \pi_{R \cup S}(\{t \mid t \in (r \times s) \text{ and } t \in (r' \times s) \text{ and } \forall A, B \in R \cap S : t[A] = t[B]\}) = \pi_{R \cup S}(\{t \mid (r \times s) \text{ and } (r' \times s) \text{ and } \forall A, B \in R \cap S : t[A] = t[B]\})\) for \(R \subseteq S\). But then \(d \text{ is the largest (in terms of cardinality) relation instance such that } d \subseteq r \times s \text{ and } \forall A, B \in R \cap S : t[A] = t[B]\). Hence we have to show \(d \bowtie s = r\). We know by definition that \(d\) is the largest (in terms of cardinality) relation instance such that \(d \times s \subseteq r \times s\) (e.g., see the textbook). But then \(d\) must be \(r\).

(11) Let \(r(R)\) and \(s(S)\) be relations where \(R \cap S = \emptyset\). Prove
\[
(r \bowtie s) \div s = r.
\]
Here \(\div\) denotes the division operator.

Since \(R \cap S = \emptyset\), \((r \bowtie s) \div s = (r \times s) \div s\). Hence we have to show that \((r \times s) \div s = r\). Let \(d = (r \times s) \div s\). We know by definition that \(d\) is the largest (in terms of cardinality) relation instance such that \(d \times s \subseteq r \times s\) (e.g., see the textbook). But then \(d\) must be \(r\).

(12) Let \(r\) be a relation on schema \(R\) and let \(s\) and \(s'\) be relations on scheme \(S\), where \(R \supseteq S\). Show that if \(s \subseteq s'\), then
\[
\begin{align*}
r \div s & \supseteq r \div s'.
\end{align*}
\]
Show the converse is false.

We use the formula for the division from problem (13)(a). Since \(s \subseteq s'\), we clearly have that \(\pi_r((\pi_r(r) \bowtie s) - r) \subseteq \pi_r((\pi_r(r) \times s) - r) = \pi_r((\pi_r(r) \bowtie s') - r)\). But then \(\pi_r(r) - \pi_r((\pi_r(r) \bowtie s) - r) \supseteq \pi_r(r) - \pi_r((\pi_r(r) \bowtie s') - r)\) and hence, \(r \div s \supseteq r \div s'\).

The converse is false. Counterexample: \(\{(r_1, s_1), (r_1, s_2), (r_2, s_1)\}, s = \{(s_1)\}, s' = \{(s_2)\}\). Now we have that \(r \div s = \{(r_1), (r_2)\}\) and \(r \div s' = \{r_1\}\), i.e., \(r \div s \supseteq r \div s'\) but not \(s \subseteq s'\).

(13) Let \(r(R)\) and \(s(S)\) be relations with \(R \supseteq S\) and let \(R' = R - S\). Note that \(t \in s\) denotes a tuple \(t\) in \(s\) and the expression \(\sigma_{S=t}(s)\) denotes the selection of exactly the tuple \(t\) on \(s\). Prove the identities

(a) \(r \div s = \pi_{R'}(r) - \pi_{R'}((\pi_{R'}(r) \bowtie s) - r)\)
By definition, \( r \div s = \{ t \mid t \in \pi_{R-S}(r) \text{ and } \forall t_s \in s \exists t_r \in r \text{ such that } \pi_S(\{t_r\}) = \{t_s\} \text{ and } \pi_{R-S}(\{t_r\}) = \{t\} \}. \)

Since \( R' = R-S \) we have that \( \pi_{R'}((\pi_{R'}(r) \bowtie s) - r) = \pi_{R'}((\pi_{R'}(r) \times s) - r). \)

Clearly, \( \pi_{R'}((\pi_{R'}(r) \times s) - r) = \{ t \mid t \in \pi_{R-S}(r) \text{ and } \exists t_s \in s \forall t_r \in r \text{ one has that } \pi_S(\{t_r\}) \neq \{t_s\} \text{ or } \pi_{R-S}(\{t_r\}) \neq \{t\} \}. \) In other words, it is the set of all “disqualified” tuples. Now, \( \pi_{R'}(r) - \pi_{R'}((\pi_{R'}(r) \bowtie s) - r) = \{ t \mid t \in \pi_{R-S}(r) \text{ and } \forall t_s \in s \exists t_r \in r \text{ such that } \pi_S(\{t_r\}) = \{t_s\} \text{ and } \pi_{R-S}(\{t_r\}) = \{t\} = r \div s. \)