Assignment 2: Mappings, induction

This assignment contains solved practice problems, numbered in red. The assigned problems and sub-problems are numbered in green.

1. (20%) Let $A = \{a, b, c\}$ and $B = \{0, 1, 2\}$. For each of the following types of mapping from A to B determine the number of possible distinct mappings of that type.

i. All mappings.

Solution. There are 9 elements (pairs) in $A \times B$, so there are $2^9 = 512$ possible mappings, i.e binary relations.

Alternative approach: For each $x \in A$ there are $2^3 = 8$ options for output-set. So altogether we have $8^3 = 512$ mappings.

ii. Partial functions, i.e. univalent mappings.

Solution. For each $x \in A$ we have four options for $F(x)$: $0, 1, 2$ and *undefined*. So there are $4^3 = 64$ partial-functions from A to B.

(a) Total-functions. [Hint: Similar to (ii)]

Solution. For each $x \in A$ we have three options for $F(x)$: $0, 1$ and 2. So there are $3 \times 3 = 9$ total-functions from A to B.

(b) Total mappings. [Hint: The alternative in (i), but $\emptyset$ can no longer be an output-set.]

Solution. For each $x \in A$ there are 7 options for the output-set, given that $\emptyset$ is excluded. So altogether we have $7^3 = 343$ total mappings.

(c) Surjective mappings. [Hint: Use (b)]

Solution. The surjective mappings from A to B are a mirror image of the total mappings from B to A . From (b) the number of such total mappings, when both domain and 5range have 3 elements, is 343.

(d) Injective mappings.

Solution. The injective mappings from A to B are a mirror image of the partial functions from B to A . From (ii) the number of those, when both domain and range have 3 elements, is 12.

(e) Bijections. [Hint: Use (d)]

Solution. Since A and B have both three elements, every injection is a surjection, i.e. a bijection. So by (d) the answer is 6.

2. (20%) Let $f: \mathbb{N} \to A$ be an injection, and B a set.

(a) Define an injection $g : \mathbb{N} \times B \to A \times B$.

Solution. Define, for $x \in \mathbb{N}$ and $y \in B$ $g(\langle x, y\rangle) = \langle f(x), y\rangle$. g is injective, because if $g(\langle x', y'\rangle) = g(\langle x, y\rangle)$ i.e. $\langle f(x'), y'\rangle = \langle f(x), y\rangle$ then $x' = x$ since f is injective and $y' = y$ by the definition of ordered pairs.

(b) Define an injection $j : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(A)$.

Solution. For $A \subseteq \mathbb{N}$ let $j(A) = \{f(n) \mid n \in A\}$. j is a total-function, since it is uniquely defined for every input A. It is injective because if $A \neq A'$, say $k \in A - A'$, then by the definition of j $f(k) \in j(A)$ but $f(k) \notin j(A')$, so $j(A) \neq j(A')$.

3. (15%) Use the CBS Theorem to show that $\{\mathsf{a}, \mathsf{b}\}^* \cong \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^*$.
[Hint: For an injection $h: \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^* \to \{\mathsf{a}, \mathsf{b}\}^*$ use two-letter codes for a,b,c. (This is analogous to the binary coding of ASCII characters.)]

Solution. We have $\{\mathsf{a}, \mathsf{b}\}^* \preccurlyeq \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^*$ since the identity function on $\{\mathsf{a}, \mathsf{b}\}^*$ is an injection into $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^*$.

Conversely, define $f: \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^* \to \{\mathsf{a}, \mathsf{b}\}^*$ by

$f(w) =_{\text{df}} w$ with each a replaced by aa, b by bb, and c by ab.

f is an injection:
For every string u the string $f(u)$ has length $2\,|u|$. So if $f(u) = f(v)$ then $|u| = |v|$, and if $u = \sigma_0 \cdot \cdots \cdot \sigma_k$ and and if $v = \tau_0 \cdot \cdots \cdot \tau_m$ then $k = m$, $f(u) = f(\sigma_0) \cdot \cdots \cdot f(\sigma_k)$, and $f(v) = f(\tau_0) \cdot \cdots \cdot f(\tau_k)$.

By the definition of f, $f(\mathsf{a}), f(\mathsf{b})$ and $f(\mathsf{c})$ are all different, so $\sigma_i = \tau_i$ for $i = 1..k$, in other words $u = v$. Thus $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^* \preccurlyeq \{\mathsf{a}, \mathsf{b}\}^*$.

$\{\mathsf{a}, \mathsf{b}\}^* \cong \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}^*$ follows by the CBS Theorem.

4. (20%) For each of the following partial-functions determine whether it is (1) total; (2) injective; (3) surjective.

(a) $f: \mathbb{R} \rightharpoonup \mathbb{R}$ where f is defined by $f(x) = +\sqrt{x}$.

Solution. The partial-function f is not total, because it is not defined for negative input.

(b) $f: \mathcal{P}(\mathbb{N}) \rightharpoonup \mathcal{P}(\mathbb{N})$ where $f(A) =_{\mathrm{df}} \mathbb{N} - A$.

Solution. f is total. It is injective: if $f(A) = f(B)$, i.e. $\mathbb{N} - A = \mathbb{N} - B$, then $A = \mathbb{N} - (\mathbb{N} - A) = \mathbb{N} - (\mathbb{N} - B) = B$. It is surjective: for every $A \subseteq \mathbb{N}$ we have $A = \mathbb{N} - (\mathbb{N} - A) = f(\mathbb{N} - A)$.

(c) $f: A \rightharpoonup A$ where A is the set of living people and $f(x) =_{\mathrm{df}} x$'s oldest child.

Solution. Not total: not every person has children. Not injective: A person is often the oldest child of both their parents. Not surjective: A person need not be the oldest child of anyone.

i. $f: \mathbb{N} \rightharpoonup \mathbb{N}$ where $f(x) =_{\mathrm{df}} x$'s smallest divisor > 1. For example, $f(10) = 2$, $f(11) = 11$.

Solution. Not total: not defined for 1. Not injective: 2 is the smallest divisor of every even number. Not surjective: Only prime numbers are obtained.

A. Prove that $2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

Solution. Base. For $n = 0$ we have $2^0 = 1 = 2^{0+1} - 1$.

Step. Suppose the equation for $n = k$: $2^0 + 2^1 + \cdots + 2^k = 2^{k+1} - 1$. Then for $n = k + 1$ we have

$$\begin{aligned} 2^0 + 2^1 + \cdots + 2^n &= (2^0 + \cdots + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \qquad \text{(IH)} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{(k+1)+1} - 1 \\ &= 2^{n+1} - 1 \end{aligned}$$

By induction on $\mathbb{N}$ it follows that $2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

5. (10%) Prove by induction on $\mathbb{N}$ that for all $n \in \mathbb{N}$

$$1 + 3 + \cdots + (2n + 1) = (n+1)^2$$

Solution. **Base:** If $n = 0$ then $1 + \cdots + (2n + 1) = 2n + 1 = 1 = (0 + 1)^2$.

Step: Suppose that the identity is true for $n = k$.
Then for $n = k+1$ we have

$$\begin{aligned} 1 + 3 + \cdots + (2n + 1) &= 1 + 3 + \cdots + (2k+1) + (2k+3) \\ &= (k + 1)^2 + (2k+3) \qquad \text{(by IH)} \\ &= (k + 1)^2 + 2(k+1) + 1 \\ &= [(k + 1) + 1]^2 \\ &= (n + 1)^2 \end{aligned}$$

6. (15%) Prove by Shifted Induction that for every natural number $n \geqslant 8$ there are $a, b \in \mathbb{N}$ such that $n = 3a + 5b$. [Hint: For the induction step, you assume $k = 3a + 5b$, and you want to prove that there are a', b' such that $k+1 = 3a' + 5b'$. Consider first the case where $b' = 0$.]

Solution. Base. For $n = 8$ we can take $a = b = 1$.

Step. Suppose the given property holds for $n = k \geqslant 8$, that is $k = 3a + 5b$ for some $a, b \in \mathbb{N}$. If $b \geqslant 1$ then $k + 1 = 3(a + 2) + 5(b - 1)$. Otherwise, i.e. $b = 0$, we have $k = 3a \geqslant 8$, so $a \geqslant 3$. We have then $k + 1 = 3(a - 3) + 5 \cdot 2$.

By shifted induction it follows that for every natural number $n \geqslant 8$ there are $a, b \in \mathbb{N}$ such that $n = 3a + 5b$.

B. A ***multi-set*** is like a set but with repetition being counted. So $\{a, b\}$, $\{a, a, b\}$ and $\{a, b, b\}$ are different, and of sizes 2,3 and 3.

Show that for all $n > 0$: if R is a multi-set of size n whose elements are positive real numbers whose product $\prod R$ is 1, then its sum $\sum R \geqslant n$. [Hint: If R is of size $k+1$, with a the smallest element and b the greatest, replace a and b by their product ab; observe that $a \geqslant 1 \geqslant b$.]

Solution. Proof by induction shifted to 1.
Base. If R is a multi-set with one element a , then $a = 1$ since $\prod R = 1 \geqslant 1$.

Step. Assume the claim holds for multi-sets of size k. Let R be a multi-set of $k+1$ whose product is 1. Choose $a = \min(R)$ and $b = \max(R)$ that are distinct (though possibly the same number); this is possible because $k+1 \geqslant 2$. By choice of a, b we have $a \leqslant 1 \leqslant b$.

The multi-set $Q =_{\text{df}} R - \{a, b\} \cup \{a \cdot b\}$ has k elements.

Also, $b(1-a) \geqslant 1-a$ and therefore $b - ab + a \geqslant 1$. Put together,

$$\begin{aligned} \sum R &= (\sum Q) + (a+b-ab) \\ &\geqslant k + (a+b-ab) && \text{(IH)} \\ &\geqslant k+1 \end{aligned}$$

completing the induction.

The statement above implies that the geometric mean of a multi-set of reals is $\leqslant$ its arithmetic mean, a remarkable feat.