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Assignment 7: Dual-Clipping, PDAs

In this assignment, **"construct directly a PDA"** means that no auxiliary stack symbols are used other than the bottom-marker. This implies that you cannot construct your PDA by converting a CFG to it.

1. (15+15%) Show that the following languages over $\Sigma = \{a, b\}$ are not CF.

i. $L = \{a^i\, b\, a^i\, b\, a^i \mid i \geqslant 0\}$

Solution. *Solution phrased as failure of the dual-clipping property:*
Given $k > 0$, let $w = b^k a b^k a b^k$. We have $w \in L$ and $|w| > k$.

If p is a substring of w of length $\leqslant k$ then p intersects at most two of the three substrings b^k, so clipping letters our of p either removes an a, and the resulting string would not be in L, or removes at least one b, but not from all three blocks b^k, yielding also a string not in L. Thus L fails the dual-clipping property, and cannot be CF.

Same solution via the dual-clipping theorem:
Suppose L is generated by a CFG G. Let m be the number of variables in G, d the degree of G, and $k = d^m$.
Take $w = b^k a b^k a b^k$. Since $w \in L$ and $|w| > k$ it follows by the Dual-Clipping Theorem that there is a substring p of w, of length $\leqslant k$, so that the string w' obtained from w by removing some letters in p is also in L.
But since p has length $\leqslant k$ it intersects at most two of the blocks b^k. So clipping letters out of p either removes an a, and the resulting string would not be in L, or removes some b's from some but not all three blocks b^k, yielding also a string not in L. Thus $w' \notin L$, contradicting the assumption that G is a CFG generating L.

(a) $L = \{a^i b^j a^i \mid j > i\}$.

(b) $L = \{a^i b^j a^i b^j \mid i, j \geqslant 0\}$.

A. **i.** Construct directly a PDA that recognizes the language $L = \{\mathtt{a}^i \mathtt{c} (\mathtt{ab})^i \mid i \geqslant 0\}$.

Solution. L is recognized by the following PDA M. M pushes `a`'s on the stack as long as it reads `a`'s. On reading `c` M switches to a pair of states that pops an `a` for every `ab` read:

- States: $\{s, q, p_0, p_1, f\}$, initial state s, accepting state f.
- Transition rules:

$$s \xrightarrow{\epsilon\,(\epsilon \to \$)} q \qquad p_0 \xrightarrow{a\,(a \to \epsilon)} p_1$$
$$q \xrightarrow{a\,(\epsilon \to a)} q \qquad p_1 \xrightarrow{b\,(\epsilon \to \epsilon)} p_0$$
$$q \xrightarrow{c\,(\epsilon \to \epsilon)} p_0 \qquad p_0 \xrightarrow{\epsilon\,(\$ \to \$)} f$$

ii. Give an accepting computation trace of your PDA for `acabab`.

Solution.

$$s \xrightarrow{\epsilon\,(\epsilon \to \$)} q \qquad p \xrightarrow{\mathtt{b}\,(\epsilon \to \mathtt{b})} p \qquad r \xrightarrow{\epsilon\,(\epsilon \to \epsilon)} t$$
$$q \xrightarrow{\mathtt{a}\,(\epsilon \to \mathtt{a})} q \qquad p \xrightarrow{\epsilon\,(\epsilon \to \epsilon)} r \qquad t \xrightarrow{\mathtt{b}\,(\mathtt{a} \to \epsilon)} t$$
$$q \xrightarrow{\epsilon\,(\epsilon \to \epsilon)} p \qquad r \xrightarrow{\mathtt{a}\,(\mathtt{b} \to \epsilon)} r \qquad t \xrightarrow{\epsilon\,(\$ \to \epsilon)} f$$

iii. Give a CFG G that generates L.

Solution. $S \to \mathtt{a}\, S\, \mathtt{ab} \mid \mathtt{c}$

iv. Convert G into another PDA that recognizes L.

Solution. With initial state s and accepting state f:

$$s \xrightarrow{\epsilon\,(\epsilon \to S\$)} q \qquad q \xrightarrow{\epsilon\,(S \to \mathtt{c})} q \qquad q \xrightarrow{\mathtt{b}\,(\mathtt{b} \to \epsilon)} q$$
$$q \xrightarrow{\epsilon\,(S \to \mathtt{a}S\mathtt{ab})} q \qquad q \xrightarrow{\mathtt{a}\,(\mathtt{a} \to \epsilon)} q \qquad q \xrightarrow{\epsilon\,(\$ \to \epsilon)} f$$

2. (10+5+10%)

(a) Construct directly a PDA that recognizes the language

$L = \{\mathtt{a}^{p+q}\mathtt{b}^{q}\mathtt{c}^{p} \mid p,q \geqslant 0\}$.

(b) Give an (accepting) computation trace of your PDA for **aaabcc** , and a (non-accepting) trace for **abc**.

(c) Define a CFG that generates L, and then convert it into another PDA N that recognizes L, different from the one in (a).

3. (25%) Construct directly a PDA recognizing the language

$L = \{\mathtt{a}^{i+j}\mathtt{b}^{j+k}\mathtt{c}^{k+i} \mid i,j,k \geqslant 0\}$.

4. (20%) Let $\Sigma = \{\mathtt{a},\mathtt{b},\mathtt{c}\}$, and let $X, Y \subseteq \Sigma^*$ be CFL's.

Prove that the following languages are also CF.

i. $L = \{ u \cdot v \mid u \in A,\ v \in B,\ u$ without $\mathtt{a}$'s and v without $\mathtt{b}$'s $\}$.

Solution. Let $X' = X \cap \{\mathtt{b},\mathtt{c}\}^*$ and $Y' = Y \cap \{\mathtt{a},\mathtt{c}\}^*$. Thus $L = X' \cdot Y'$. X' is a CFL as the intersection of a CFL and a regular language, and similarly for Y'. So L is a CFL as the concatenation of two CFL's.

ii. $L = \{ x \cdot y \cdot x' \mid x, x' \in X,\ y \in Y \}$.

Solution. $L = X \cdot Y \cdot X$, a concatenation of CFL's, and therefore a CFL.

(a) Let R be a regular language and K a CFL. Show that the following is a CFL.

$L = \{ x \cdot \# \cdot x' \mid x, x' \in R,\ |x| = |x'| \}$ (where $\#$ is a fresh symbol).

(b) For R and K as above. $L = \{ x \cdot y \cdot x' \mid x, x' \in R,\ y \in K,\ |x| = |x'| \}$.