Assignment 8: Decidability and reductions

This assignment contains solved practice problems, numbered in red.
The assigned problems and sub-problems are numbered in green.

1. (5%)

i. Prove that the concatenation of decidable languages is decidable.

Solution. If L and K have decision algorithms, then decide whether $w \in L \cdot K$ by cycling through all partitions $w = u \cdot v$, and checking, using the given decision algorithms for L and K, whether $u \in L$ and $v \in K$. If both are true for some u, v then stop and accept. If not, reject.

(a) Prove that the star of a decidable language is decidable.

Prove that every regular language is decidable. [Hint: Use the closure of the collection of decidable languages under set operations. Alternatively, explain how a DFA can be construed as an algorithm.]

2. (5+10+10%)

(a) Exhibit two disjoint undecidable languages whose union is decidable.

(b) Show that if $D, N \subseteq \Sigma^*$ are disjoint, where D is decidable and N undecidable, then the union $D \cup N$ is undecidable.

(c) Exhibit two undecidable languages who intersection is infinite but decidable. [Hint: Consider $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, undecidable Σ-languages L, K and decidable Σ-language D. What about $\mathtt{a} \cdot D \cup \mathtt{b} \cdot L$ and $\mathtt{a} \cdot D \cup \mathtt{c} \cdot K$?]

3. (10+5+5%) Let $L \subseteq \{\mathtt{a}, \mathtt{b}\}^*$, and define $X = \mathtt{a} \cdot L \cup \mathtt{b} \cdot \bar{L}$

(a) Prove that if L is decidable then so is X.

(b) Prove that $\bar{L} \leqslant_c X$.

(c) Conclude that if X is decidable then so is $\bar{L}$.

4. (5%) We showed that the problem ACCEPTANCE is SD but not decidable. Show that its complement is not even SD.

5. (15%) Show that every infinite SD language has an infinite decidable sub-language. [Hint: An infinite SD language is computably enumerated, and from its computable enumeration we can extract an orderly enumeration of a sub-language.]

i. Show that there is an infinite language L without any infinite decidable sub-language. [Hint: Explain why there is a listing $L_1, L_2, \ldots$ of all infinite decidable languages $L \subseteq \{0,1\}^*$. Now define $L = \{w_0, w_1, w_2 \ldots\}$ as follows. Let $w_0 = \varepsilon$; and given w_i, let u be the first string in L_i longer than w_i, and take w_{i+1} to be a string longer than u. Why is L infinite? Why can't we have $L_i \subseteq L$ for any i?]

Solution. The collection of Turing deciders is countable (when no computability condition is required). So, by elementary Set Theory, there is a listing $L_1, L_2, \ldots$ of all infinite decidable languages $L \subseteq \{0,1\}^*$.

Define $L = \{w_0, w_1, w_2 \ldots\}$ as follows. Let $w_0 = \varepsilon$; and given w_i let u_i be the first string in L_i longer than w_i. Such a u_i must exist, since L_i is infinite. Take w_{i+1} to be any string longer than u_i. So $|w_{i+1}| > |u_i| > |w_i|$. By definition, $u_i \in L_i$. But $u_i \notin L$, because the longest string in L of length $\leqslant |u_i|$ is w_i, which is shorter than u_i. Since $u_i \in L_i - L$ it follows that $L_i \not\subseteq L$ for all $i \geqslant 1$.

6. (15%) Let $L \subseteq \Sigma^*$ and define $D =_{\text{df}} \{ww \mid w \in L\}$.

(a) Show that $L \leqslant_c D$.

i. Assume that $\varepsilon \notin L$. Show that $D \leqslant_c L$.

Solution. Define

$$\rho(x) = \textbf{if } x \text{ is of the form } ww \textbf{ then } w \textbf{ else } \varepsilon.$$

Clearly, ρ is computable.

If $x \in D$ then, by the definition of D, $x = ww$ for some $w \in L$. By the definition of ρ this implies that $\rho(x) = w$, so $\rho(x) \in L$.

Conversely, if $\rho(x) \in L$ then, since $\varepsilon \notin L$, x is the form ww and $\rho(x) = w$. But we have $ww \in D$ only if $w \in L$, so $\rho(x) = w \in L$.

7. (15%) Let E be the set of acceptors that accept every even-length string, and D the set of accepters over $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ that accept every odd-length string. Construct a computable reduction $\rho : D \leqslant_c E$.