B501, Fall 2023


# Assignment 9: Undecidability

This assignment contains solved practice problems, numbered in red.
The assigned problems and sub-problems are numbered in green.

**1.** (10%) Which of the following is true? Explain!

**(a)** There are two undecidable languages whose intersection is decidable.

**(b)** There are two undecidable languages whose union is finite.

**i.** There are two undecidable languages whose concatenation is decidable.
**Solution.** True. Let $U \subseteq \Sigma^*$ be an undecidable language. Take $X = U \cup \{\varepsilon\}$ and $Y = \bar{U} \cup \{\varepsilon\}$ . Then both $X$ and $Y$ are undecidable, but $X \cdot Y = \Sigma^*$, because any $w \in \Sigma^*$ is either in $U$ , and then $w = w \cdot \varepsilon \in X \cdot Y$, or in $\bar{U}$, and then $w = \varepsilon \cdot w \in X \cdot Y$.

**(c)** There are two decidable languages whose concatenation is undecidable.

**2.** (25%) For each of the following problems about Turing acceptors determine whether it is decidable, SD but not decidable, or not SD. You may use any method, including Rice's Theorem and Shapiro's Theorem.

**i.** Does a given acceptor $M$ accept the string `01`?
**Solution.** This is a non-trivial scope property, so it is undecidable by Rice's Theorem.
The relation $\vdash$, where $c \vdash M^\#$ iff $c$ is an accepting trace of $M$ for input `01`, is a decidable certification for this problem, so it is SD.

**ii.** Does a given acceptor $M$ accept at least two different strings?
**Solution.** This is a non-trivial scope property, so it is undecidable by Rice's Theorem.
The relation $\vdash$, where $c \vdash M^\#$ iff $c$ is a pair of accepting traces of $M$ for two different input strings, is a decidable certification for this problem, so it is SD.
(Note that the two strings on their own are ***not*** an adequate certificate, because checking that they are accepted may not be decidable!)

**(a)** Does a given Turing acceptor $M$ accept $\varepsilon$ within $10^{10}$ steps?

**(b)** Does a given Turing acceptor $M$ accept ***some*** string within $10^{10}$ steps?
[Hint: How large are the strings that $M$ can actually read within $10^{10}$ steps?]

**(c)** Does a given Turing acceptor $M$ accept fewer than 10 different strings?
Hint: You may invoke Shapiro's Theorem.]

**(d)** Does given Turing acceptor $M$ recognize a decidable language? [Hint: Shapiro's Theorem]

**(e)** Does a given acceptor $M$ accept any string? [Caution: It is undecidable whether $M$ accepts a string $w$.]

**A.** Prove that if $L \subseteq \Sigma^*$ has a ***semi-decidable*** certification, then $L$ is SD.

**Solution.** Suppose $\vdash_L$ is a SD certification for a language $L \subseteq \Sigma^*$. The relation $\vdash_L$ being SD, there is a decidable certification $\vdash_{cert}$ for it. Consider the decidable binary relation $\vdash$ that holds between a string $d\#c$ and $w \in \Sigma^*$ iff $d \vdash_{cert} \langle c, w \rangle$. ( $\# \notin \Sigma$). Since $\vdash_{cert}$ is a certification for $\vdash_L$, we have $d\#c \vdash w$, i.e. $d \vdash_{cert} (c, w)$. iff $c \vdash_L w$, i.e. iff $w \in L$.

**3.** (25%) Let $L \subseteq \Sigma^*$, where $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Prove:

**i.** $L$ is decidable iff $L = \{w \mid f(w) = \varepsilon\}$ for some computable function $f : \Sigma^* \to \Sigma^*$.

**Solution.** If $L = \mathcal{L}(M)$ for some decider $M$ then the following function $f : \Sigma^* \to \Sigma^*$ is computable. On input $w$ $f$ outputs $\varepsilon$ if $M$ accepts $w$ and $\mathtt{a}$ otherwise. Then $L = \{w \mid f(w) = \varepsilon\}$.

Conversely, if $L = \{w \mid f(w) = \varepsilon\}$ where $f : \Sigma^* \to \Sigma^*$ is computable by a transducer $T$, then $L$ is recognized by an acceptor that, on input $w$, simulates $T$ on $w$ and accepts if and when an output $\varepsilon$ is obtained.

**(a)** $L$ is SD iff $L = \{w \mid f(w) = \varepsilon\}$ for some computable ***partial***-function $f : \Sigma^* \rightharpoonup \Sigma^*$.

**(b)** We know that a language $L \subseteq \Sigma^*$ is recognized iff it is computably enumerated, i.e. is the target of a computable functions $f : \mathbb{N} \to grS^*$.

Prove that the same remains true if we take for $f$ computable ***partial-***functions.

**4.** (20%) The problem COMMON-ACCEPT asks whether a given pair $(M_0, M_1)$ of Turing acceptors accept a common string.

**(a)** Prove that COMMON-ACCEPT is SD.

**(b)** Define a computable reduction of $\varepsilon$-ACCEPT to COMMON-ACCEPT. Conclude that COMMON-ACCEPT is not decidable. (Rice Theorem, as stated, cannot be used here directly, because COMMON-ACCEPT is about two acceptors.)

**5.** (20%) The problem SUBLANGUAGE asks whether a given pair $(M, M')$ of Turing acceptors satisfies $\mathcal{L}(M) \subseteq \mathcal{L}(M')$.

**i.** Define a computable reduction of $\varepsilon$-ACCEPT to SUBLANGUAGE.

**Solution.** Fix an acceptor $E$ for the singleton language $\{\varepsilon\}$. Let $\rho$ be a function that maps an instance $M$ of $\varepsilon$-ACCEPT to the instance $\langle E, M\rangle$ of SUBLANGUAGE. $\rho$ is clearly computable, as a purely syntactic program modification. $M$ accepts $\varepsilon$ iff $\{\varepsilon\} \subseteq \mathcal{L}(M)$, that is iff $\rho(M^{\#}) = \langle E, M\rangle \in$ SUBLANGUAGE, so $\rho$ is a reduction.

**(a)** Define a computable reduction of $\varepsilon$-NONACCEPT to SUBLANGUAGE.

**(b)** Conclude that neither SUBLANGUAGE nor its complement are SD.