SETS

RELATIONS, MAPPINGS, SIZE

What are sets

- A $\boxed{\textit{set}}$ is ***a collection into a whole of some well-recognized objects,*** dubbed the set's $\boxed{\textit{elements.}}$
- We write $a \in S$ for "a *is an element of* S"

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- **Shouldn't concepts be defined using previously defined ones?**

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- The concept of *set* is "defined" here in terms of "collection" and "whole", i.e. synonyms of "set"!
- **Shouldn't concepts be defined using previously defined ones?**
- Regressing this way cannot go on indefinitely: we must stop with concepts that are left ***undefined***.
- We only explain those informally,
  hoping to establish some
  shared imagery, intuitions and understanding.
  "Set" is just such a concept.

Exhibiting sets

- Sets are determined by their elements.
  That is, if sets A and B have the same elements,
  then they are one and the same set,
  even if they are described in very different ways.
- This is the $\boxed{\textit{Principle of Extensionality}}$.

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- Sets are determined by their elements.
  That is, if sets A and B have the same elements,
  then they are one and the same set,
  even if they are described in very different ways.
- This is the ***Principle of Extensionality***.
- It implies that finite sets can be defined
  by exhibiting their elements: $\{a_1, \ldots, a_k\}$.
  So $\{0, 1\}$, $\{1, 0\}$ and $\{0, 0, 1\}$ are all the same set.

Names and notations for special sets

- Some sets are commonly assumed as given, and assigned notations.
  - For an alphabet Σ, the set Σ^* of Σ-strings.

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  - ***nat*** or $\mathbb{N}$: The set of natural numbers $0, 1, 2, 3\ldots$.
  - ***int*** or $\mathbb{Z}$: The integers
  - $\mathbb{Q}$: the rational numbers (Q for "quotients")
  - $\mathbb{R}$: the real numbers (the "real number line")

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  - $\mathbb{R}$: the real numbers (the "real number line")
  - The ***empty set,*** denoted $\emptyset$, which has no elements.
- A set with exactly one element, however complex, is a ***singleton.***
  Examples: $\{0\}$, $\{\emptyset\}$, $\{\{\emptyset\}\}$ and $\{\mathbb{N}\}$

Abstraction notation

- Another approach to defining sets is to delineate them
  by certain properties, as in
  "the set of registered voters".

- Such definitions are captured by the notational convention
  $\{x \mid$ a property of $x\}$.

- Between braces: (1) a declared variable, say x ,
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- A set's elements can themselves be complex entities!
  Examples: $\{\emptyset\}$, $\{\mathbb{N}\}$, $\{\emptyset, \{\emptyset\}\}$.

Conventions for numeric intervals

- To denote intervals of integers or real numbers we indicate end-point's inclusion with a bracket, and exclusion with a parenthesis.

$$(p..q) \;=\; \{x \mid p < x < q\} \qquad \text{(\textbf{\textit{open interval}})}$$

$$[p..q] \;=\; \{x \mid p \leqslant x \leqslant q\} \qquad \text{(\textbf{\textit{closed interval}})}$$

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Relations between sets

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  - For any set A: $A \subseteq A$ and $\emptyset \subseteq A$.
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- If $A \subseteq B$ and $B \subseteq A$
  then A and B have the same elements.
  By Extensionality this implies $A = B$.

True or false?

$0 \in \{0, 1\}$	$\mathbb{N} \subseteq \{\mathbb{N}\}$
$\{0\} \subseteq \{0, 1\}$	$\mathbb{N} \in \{\mathbb{N}\}$
$\{0\} \in \{0, 1\}$	$\emptyset \subseteq \{\emptyset\}$
$\{0, 1, 1\} \subseteq \{1, 0\}$	$\{\emptyset\} \subseteq \emptyset$
$\{0, 1\} \subseteq \mathbb{N}$	$\emptyset \in \emptyset$
$\{0, 1\} \subseteq \{\mathbb{N}\}$	$\emptyset \in \{\emptyset\}$

The perils of abstraction

- In the template $\{x \mid \cdots x \cdots\}$,
  does x stand for "anything"?

- If that were so, we'd be able to define

$$R =_{\text{df}} \{x \mid x \notin x\}$$

  That is, for all x

$$x \in R \quad \textbf{IFF} \quad x \notin x$$

- In particular, if we take x to be R then

$$R \in R \quad \textbf{IFF} \quad R \notin R$$

  A contradiction!

- This is known as $\boxed{\textit{Russell's Paradox.}}$

The Separation Principle

- There is a circularity at the root of the definition of R:

  "all sets" includes the set R itself,

  which is defined in terms of "all sets."

- Work-around: Zermelo's **_Separation Principle:_**

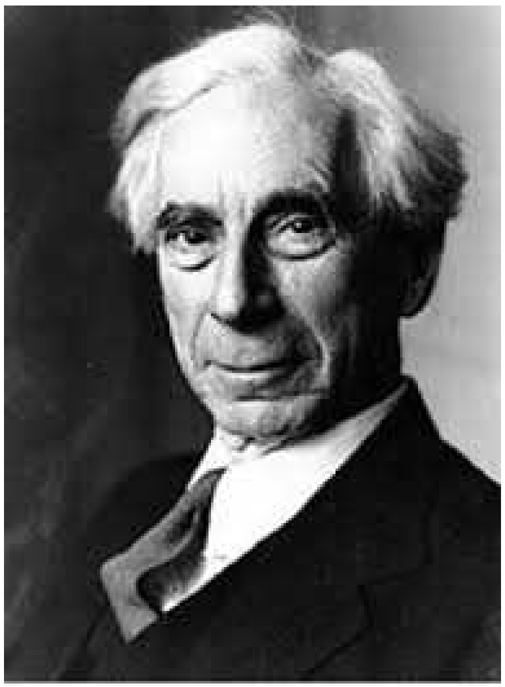
  For a *given* set S we may define $\{x \in S \mid \cdots x \cdots\}$.

  We "separate" out the elements of S along the given property.

- This blocks Russell's paradox:

  S would have to be "all sets", which is not admissible as a set.

Bertrand Russell and Ernst Zermelo



Russell (1872-1970)

Zermelo (1871-1953)

The Diagonal Method

- Russell's Paradox epitomizes a powerful line of reasoning.

  To illustrate, let's call a book ***modest*** if its text does not mention its title.

  Question: Can we compile a catalog of all modest books?

- Suppose such a catalog existed, with title $\boldsymbol{M}$ say.

  A book is listed in $\boldsymbol{M}$ iff it does not mention itself.

  In particular, $\boldsymbol{M}$ is listed in $\boldsymbol{M}$ iff $\boldsymbol{M}$ is not listed in $\boldsymbol{M}$.

- Consequence: ***There can be no catalog of all modest books!***

- **Where does the contradiction come from?**

Contradictions via two-faced objects

- The catalog argument refers to each book in two ways:

  as a title, and as contents.

- Russell's Paradox refers to each set in two ways:

  as a set of other objects, and as a possible element of other sets.

- This duality is the core of the ***Self-reference Method***

  AKA the ***Diagonal Method.***

  (A matrix's diagonal is where row $\#i$ meets column $\#i$.)

- This duality is ingrained in computing:

  a program is both a string and an algorithm.

Operations on sets

- $$\begin{aligned} A \cap B &= \{x \mid x \in A \text{ and } x \in B\} \\ A \cup B &= \{x \mid x \in A \text{ or } x \in B\} \\ A - B &= \{x \mid x \in A \text{ and } x \notin B\} \end{aligned}$$

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- When all sets considered are subsets of some set U ,

  we refer to $U - A$ as the **complement** of A,

  and write $\bar{A}$ for it.

$\cup$ *is the dual of* $\cap$

- We have $\overline{A \cap B} = \bar{A} \cup \bar{B}$:

$$x \notin A \cap B \text{ \textbf{\textit{iff}} } x \notin A \text{ or } x \notin B$$

"not both true" is the same as "at least one is false"

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- If a finite A has n elements,

  then $\mathcal{P}(A)$ has 2^n elements:

  - A subset $B \subseteq A$, is fixed by choosing, for each $x \in A$,

    whether or not $x \in B$.

  - Each choice doubles the number of previous choices.

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- More generally, a ***collection*** C of sets is $\boxed{\textit{disjoint}}$ if

  $A \cap B = \emptyset$ for every distinct $A, B \in C$.

  (The phrase ***pairwise-disjoint*** means the same thing.)

Disjoint sets

- Sets A, B are $\boxed{\textit{disjoint}}$ if $A \cap B = \emptyset$,

  i.e. they have no element in common.

- Example:

  The Canadian citizenry is disjoint from the Japanese citizenry

  (Japan disallows dual citizenship...)

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- Example: The collection of open intervals $(0..1)$, $(1..2)$, $(2..3)$, $(3..4)$, $\cdots$

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- Non-example:

  - English words fall into eight parts of speech,

    but this is not a partition: some words are both noun and verb.

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  - Classify humanity by birth-year:

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  - Classify $\mathbb{R}$ into the half-closed intervals

    $[n..n+1)$, (n an integer).

RELATIONS

Ordered pairs

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  we can form the $\boxed{\textit{ordered-pair}}$ $\langle a, b \rangle$.

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- As we did for sets, we take the formation of ordered-pairs

  and ordered tuples to be a basic, intuitively clear, operation.

Set-product

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  - $\mathbb{R} \times \mathbb{R}$ is the real-number ***plane***.
  - $\mathbb{Z} \times \mathbb{Z}$ is the integer grid.
  - $\langle$ US town-names$\rangle$ $\times$ $\langle$ US state-names$\rangle$.

  Some elements: $\langle$*Bloomington*, *Indiana*$\rangle$, $\langle$*Cambridge*, *Ohio*$\rangle$, $\langle$*Portland*, *Maine*$\rangle$

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- With few exceptions we use the usual ***infix notation:*** For $\langle a, b\rangle \in R$ we write $a\,R\,b$.

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- Reporting relation in an organization.
- Dependency relation between components of software modules.

⋆ *Renatus Cartesius*

- René Descartes, 1596-1650
- `https://en.wikipedia.org/wiki/Ren%C3%A9_Descartes`
- The unity of Mathematics!

Visual representation by di-graphs

- Any binary relation $R \subseteq A \times A$

  can be represented as a directed-graph without multiple edges:

  The vertices are the elements of A

  and there is an edge $x \leftrightarrow y$ iff $x(R)y$.

MASQUERADING AS EQUALITY

Reflexive relations

- One useful type of relations consists of those who share the essential properties of equality.
- $R \subseteq A \times A$ is $\boxed{\textbf{\textit{reflexive on } } A}$ if xRx for all $x \in A$.
- Note that this property of R, standing alone.

Examples

- Identity over a set A.

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- Inequality $<$ between real numbers

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- *has-same-prime-factors-as* (over $\mathbb{N}$)

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Which are symmetric?

- *loves*

Which are symmetric?

- *loves* Unfortunately not

Which are symmetric?

- *loves*
- *earlier-than*

Which are symmetric?

- *loves*
- *earlier-than* No

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- Such class can be identified by one of its members.

  We'll see that it does not matter which one. So we define:

- Given an equivalence $\sim$ over A, and $x \in A$,

  the $\boxed{\sim\text{-class of } x}$ is defined by

$$[x]_\sim \;=_{\text{df}}\; \{y \in S \mid y \sim x\}$$

Examples of equivalence-classes

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- Over an undirected graph, *is-connected-to*

  $[u]_\sim =$ the connected component of u

Class-naming is robust

- What if we "name" $[a]_\sim$ by a different a' in it?
- The choice of "name" makes no difference!:

  if $a' \in [a]_\sim$ then $[a']_\sim = [a]_\sim$.

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$$
\begin{aligned}
x \in [a]_\sim \quad &\Rightarrow \quad x \sim a && \text{(dfn of } [a]_\sim\text{)} \\
&\Rightarrow \quad x \sim a' && \text{(transitivity, since } a \sim a' \\
&\Rightarrow \quad x \in [a']_\sim
\end{aligned}
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- $\Rightarrow$: Suppose $a \sim a'$, show $[a]_\sim \subseteq [a']_\sim$

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- $\Leftarrow$: Suppose $[a]_\sim \subseteq [a']_\sim$ show $a \sim a'$

Class-naming is robust

- What if we "name" $[a]_\sim$ by a different a' in it?

- The choice of "name" makes no difference!:

  if $a' \in [a]_\sim$ then $[a']_\sim = [a]_\sim$.

  In fact $\quad a \sim a' \quad$ iff $\quad [a]_\sim = [a']_\sim$

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- $\Leftarrow$: Suppose $[a]_\sim \subseteq [a']_\sim$ show $a \sim a'$

$$\begin{array}{llll} a \sim a & & & \text{(reflexivity)} \\ & \Rightarrow & a \in [a]_\sim & \text{(dfn of } [a]_\sim\text{)} \\ & \Rightarrow & a \in [a']_\sim & \text{(since } [a]_\sim = [a']_\sim\text{)} \\ & \Rightarrow & a \sim a' & \text{(dfn of } [a']_\sim\text{)} \end{array}$$

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  - Natural numbers always compare under $\leqslant$,

    but not every two sets compare under $\subseteq$.
  - $\mathbb{Q}$ has an element between any two elements, $\mathbb{N}$ does not.

What is common to all order relations?

- Intuition of order is rooted in the natural order:

$$0 < 1 < 2 < 3 \cdots \quad .$$

- Its most essential features are
  - ***Asymmetry:*** uRv contradicts vRu
  - ***Transitivity:*** uRv and vRw together imply uRw.

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- Its most essential features are
  - ***Asymmetry:*** uRv contradicts vRu
  - ***Transitivity:*** uRv and vRw together imply uRw.
- But historically $\leqslant$ was considered a more useful paradigm.

  So the common characterization of "order" has shifted to be:
- A relation R over a set A is an ***order on*** A if it is
  - Reflexive on A
  - Transitive
  - ***Anti-symmetric:*** uRv and vRu together imply $u = v$.

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  $\sigma_1 \cdots \sigma_p \prec \tau_1 \cdots \tau_q$ if either $p < q$

  or $p = q$ and for some $i < p$, $\sigma_1 \cdots \sigma_i = \tau_1 \cdots \tau_i$ and $\sigma_{i+1} \prec \tau_{i+1}$
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- I.e. strings are ordered by length, and lexicographically within each length.
- Any set of strings can be listed in increasing $\prec$ order.
- This is not possible with usual lexicographic order:

  For example, the one-letter Latin string **b**

  is preceded by the infinitely many strings that start with **a**.

MAPPINGS

Binary relations as input-output processes

- A relation from A to B can often be construed as a process that takes elements of A as input and yields corresponding output-values in B.
- For example, the relation *parent-of* can be construed as yielding for any person each one of their children.
- Interpreting relations as processes is not always natural. It is awkward to construe $<$ on $\mathbb{N}$ as a process that maps each x to each $y > x$.

Mappings

- A relation $R \subseteq A \times B$ does not determine the sets A and B,
  because $R \subseteq A' \times B'$ for every $A' \supseteq A$ and $B' \supseteq B$.
- For example, if R maps people to their ancestors aged $\leqslant 150$
  then it also maps people to their ancestor aged $\leqslant 200$.
- We define a $\boxed{\textbf{\textit{mapping}}}$ as a triple (R, A, B) where $R \subseteq A \times B$.
  We write $R: \ A \Rightarrow B$ to state that (A, R, B) is a mapping.
  A is the $\boxed{\textbf{\textit{domain}}}$ of the mapping and B its $\boxed{\textbf{\textit{range}}}$.

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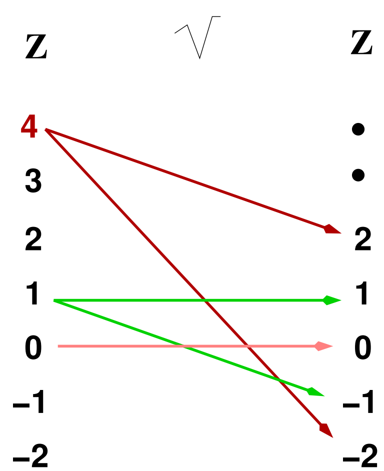
- Example: Consider the relation $\surd = \{\langle x^2, x\rangle \mid x \in \mathbb{R}\}$.
  Then $\surd[4] = \{2, -2\}$ $\surd[0] = \{0\}$ $\surd[-4] = \emptyset$



Operations on mappings

Mapping inverse

- The $\boxed{\textit{inverse}}$ of a mapping $R: A \Rightarrow B$ is the mapping $R^{-1}: B \Rightarrow A$ where $x\,(R^{-1})\,y$ iff $y\,(R)\,x$.
- The superscript -1 is borrowed from the reciprocal function $x^{-1} = 1/x$ over $\mathbb{R}$.

Examples

- Inverse of *parent-of* is *child-of*
- Inverse of *loves* is *is-loved-by*
- Inverse of *has-SSN* is *is-SSN-of*
- The inverse of $<$ is $>$,
  and the inverse of $\leqslant$ is $\geqslant$.

Inverting the inverse

- $(R^{-1})^{-1} = R$
- ***Proof.*** $x\,(R^{-1})^{-1}\,y$ iff $y\,(R^{-1})\,x$
  iff $x\,(R)\,y$

Relational-composition

- The $\boxed{\textit{relational-composition}}$ of mappings $R : A \Rightarrow B$ and $Q : B \Rightarrow C$ is the mapping $(R\,;Q) : A \Rightarrow C$ where
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- Relational-composition interprets mappings as processes,
  therefore following the procedural order.

  The semi-colon notation reflects this interpretation.

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- Between people: *mother-of* **;** *parent-of* is *grandma-of*.

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- Over subsets of $\mathbb{N}$:
  $(\subseteq)\ ;\ (\subseteq)$ is $\subseteq$

but $(\subset)\ ;\ (\subset)$ is "extending by at least 2 elements".

Inverse of a composition

- $(R\,;Q)^{-1} = Q^{-1}\,;R^{-1}$

Proof. $x\,(R\,;Q)^{-1}\,z$ ***iff*** $z\,(R\,;Q)\,x$ (dfn of inverse)

iff $z\,R\,y$ and $y\,Q\,x$ some y (dfn of $;$)

iff $y\,R^{-1}\,z$ and $x\,Q^{-1}\,y$ some y (dfn of inverse)

iff $x\,(Q^{-1}\,;R^{-1})\,z$ (dfn of $comp$)

Properties of mappings

Four input/output properties

We'll consider four properties that mappings $R\colon A \Rightarrow B$ may have.

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For every $x \in A$ there is at most one $y \in B$ such that $x\,R\,y$.

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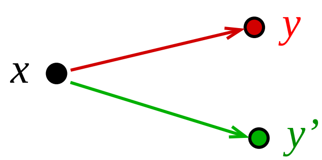

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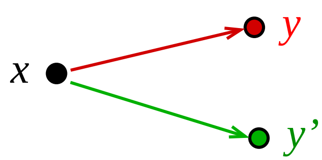

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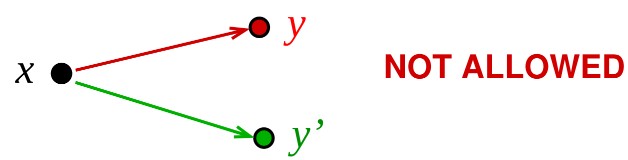

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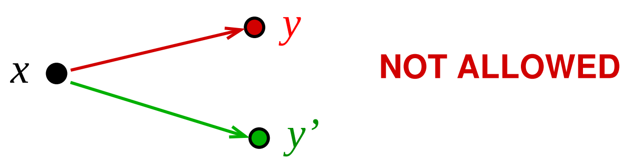

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- *married-to* is univalent assuming monogamy.

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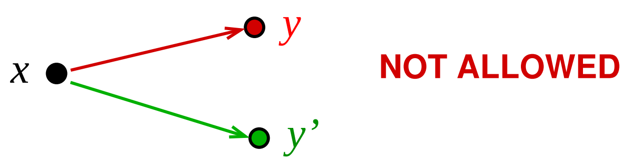

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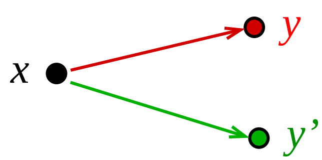

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Composition of univalent mappings

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- Then $y = y'$ because R is univalent,
  and so $z = z'$ because Q is univalent.

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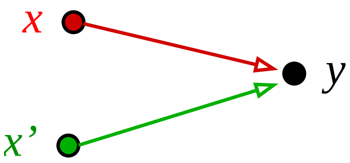

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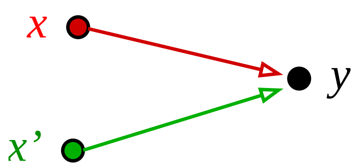

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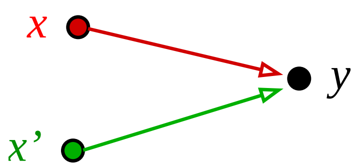

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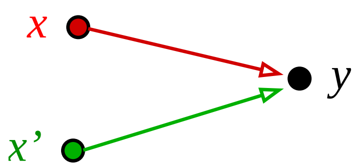

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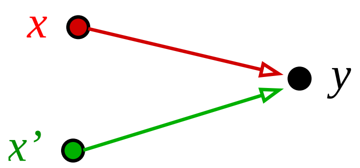

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- $\{\langle x, x^2\rangle \mid x \in \mathbb{N}\}$ is not injective: 2 and -2 are mapped to the same number.
- The mapping from people to their name is not injective: different people may have the same name.

Which are injective?

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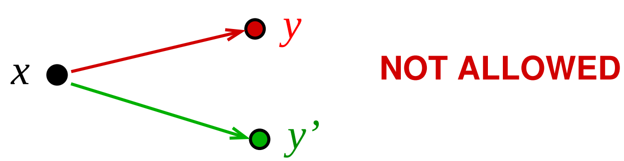
  Yes. No SSN is assigned to two different persons.

Injective is the dual of univalent

- Univalent:
  At most one output per input.



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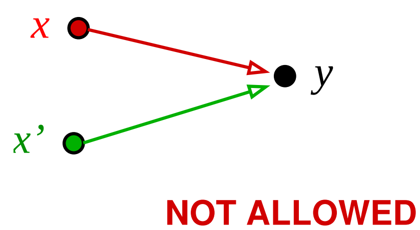

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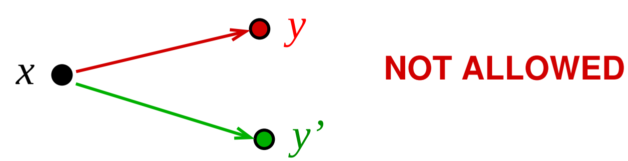

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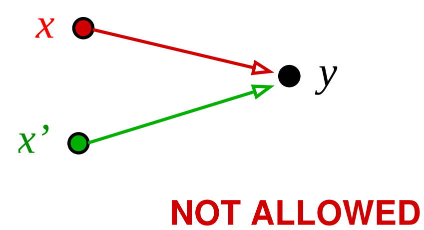

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- ***Proof.*** $x\,(R)\,y$ plus $x\,(R)\,y'$ imply $y = y'$
  iff
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- $y = y'$ because Q is injective,
  and therefore $x = x'$ because R is injective.

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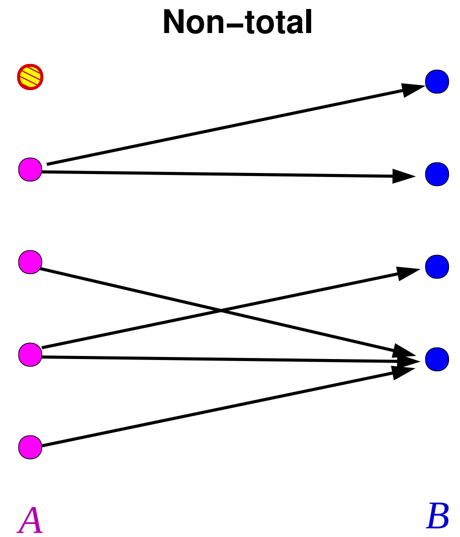

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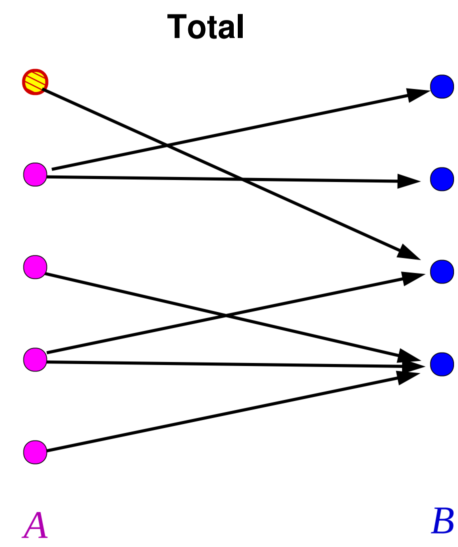

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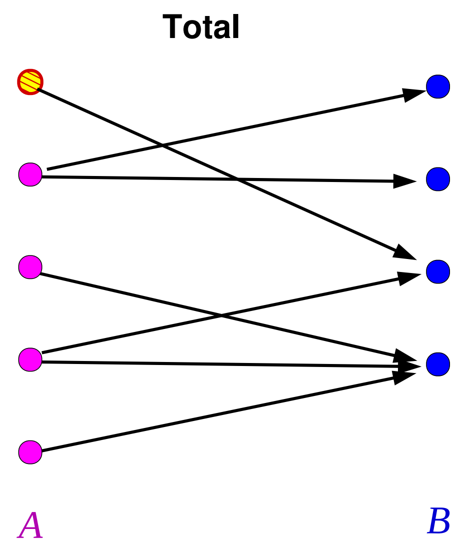

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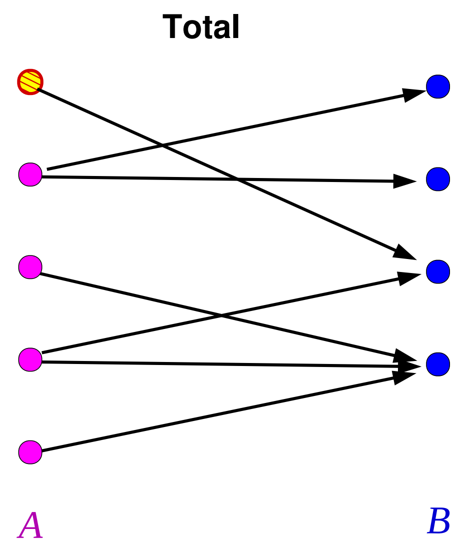

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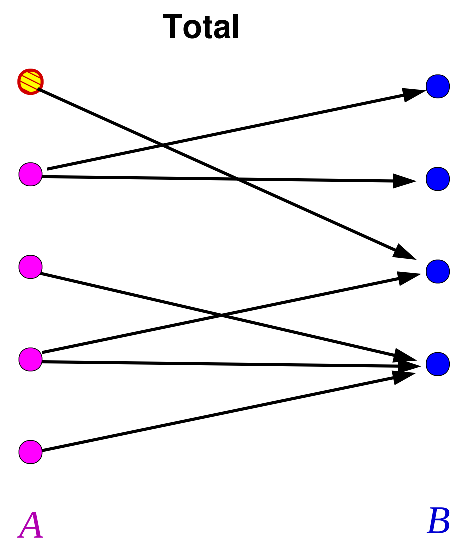

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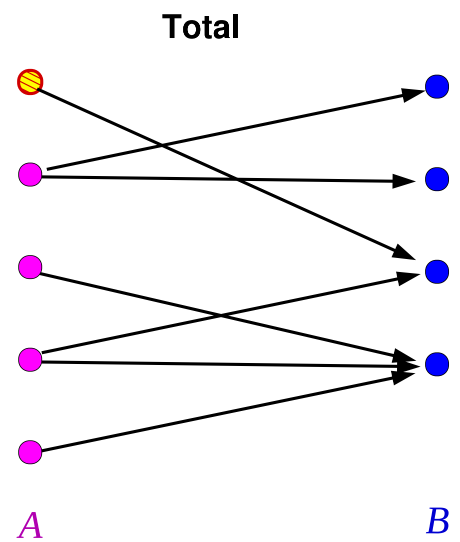

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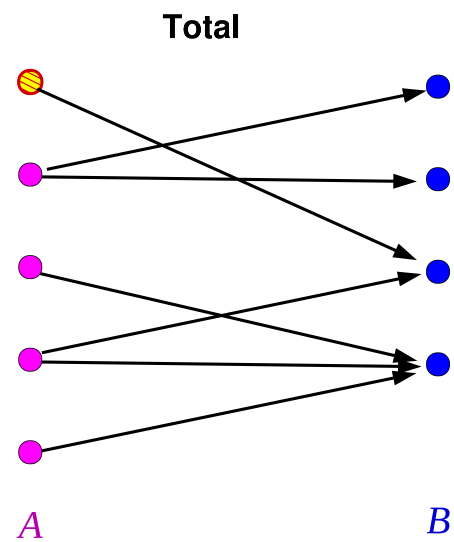

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  Put together, we obtain $x \,(R ; Q)\, z$ for some z.

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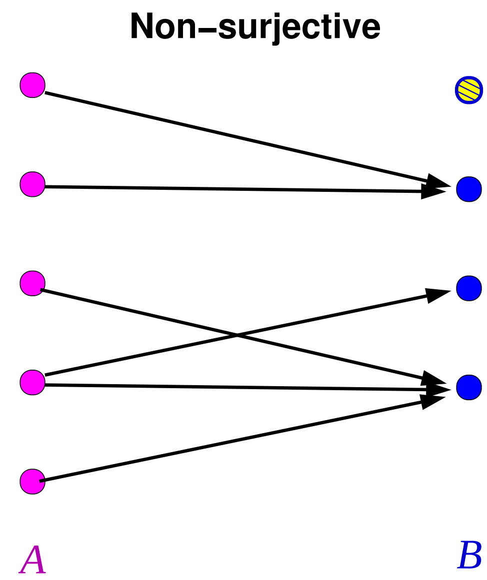

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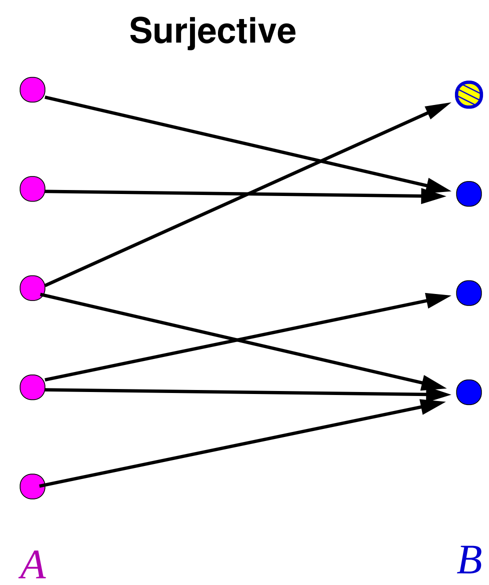

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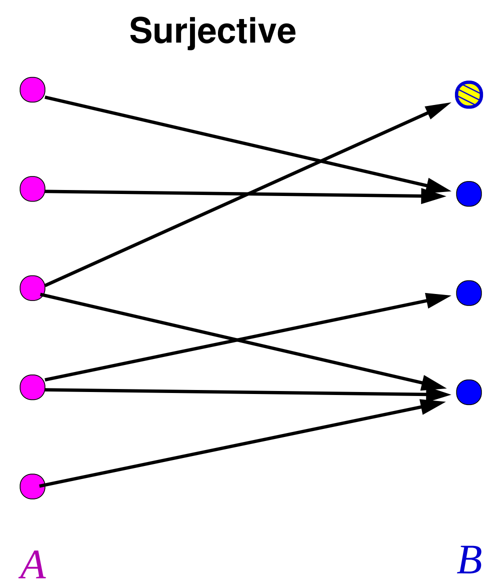

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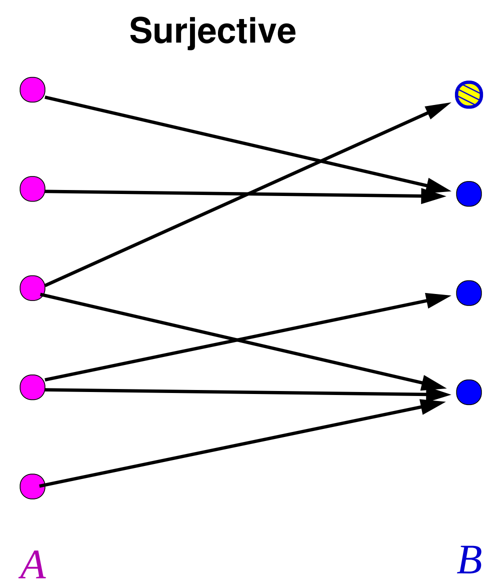

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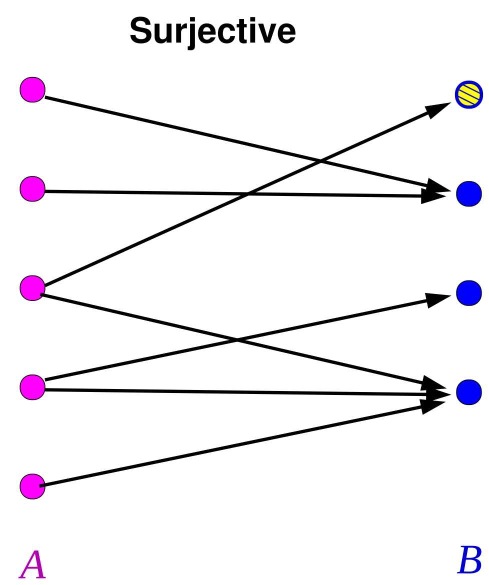

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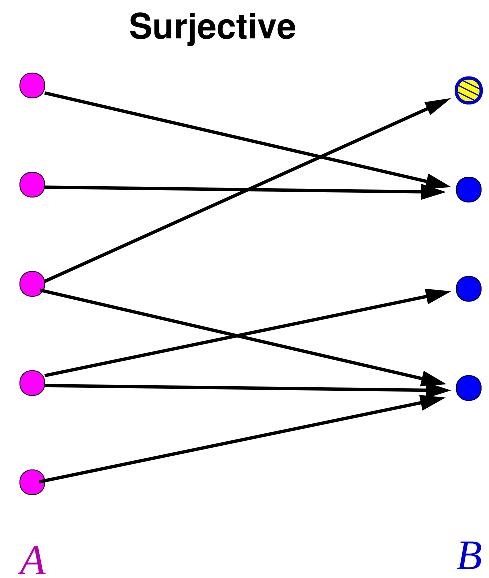

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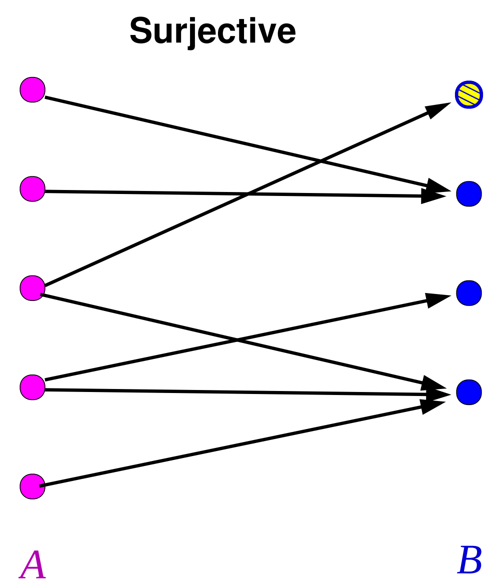

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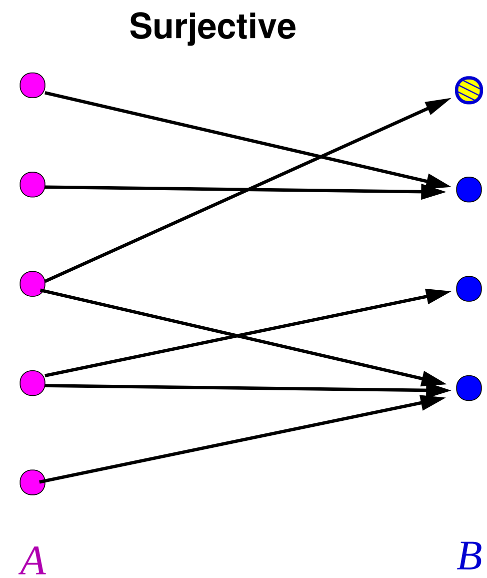

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- If $R : A \Rightarrow B$ and $Q : B \Rightarrow C$ are surjective then so is $R;Q : A \Rightarrow C$.
- ***Proof.*** Given that $Q : B \Rightarrow C$ is surjective, for every $z \in C$ there is a $y \in B$ such that $y\,(Q)\,z$.
- This implies, Since $R : A \Rightarrow B$ is surjective, that $x\,(R)\,y$ for some $x \in A$.
- Thus $x\,(R;Q)\,z$.
  Since this holds for every $z \in C$, $R;Q : A \Rightarrow C$ is surjective.

FUNCTIONS

Functions: univalent and total

- When a mapping $R: A \Rightarrow B$ is univalent
  we also say that it is a $\boxed{\textit{partial-function}}$ (from A to B),
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- A partial-function $R : A \rightharpoonup B$ is "partial" in that
  it is ***not necessarily*** total (on B).

  So every total-function is also a partial-function!

  And a partial-function may be total or ***non-total***.

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- When $F : A \rightharpoonup B$ (i.e. totality not assumed),

  we still write $F(x)$ for the y satisfying $x\,(F)\,y$,

  and say that F is ***undefined*** if no such y exists.

Explicit function definitions

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- To refer to a function on the fly, without naming it,
  we use the ***"maps-to"*** notation: $x \mapsto E$.
  Example: $x \mapsto 2x + 1$.

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- The reciprocal-function over $\mathbb{R}^+$ $x \mapsto 1/x$ is a total-function.
- $F(n) =$ the first prime number $\geqslant n$.
  That this function is total is akin to saying that
  there are infinitely many primes.

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- For any sets A, B we have an ***empty partial-function*** $\emptyset : A \rightharpoonup B$.
  That is, $\emptyset(x)$ is undefined for all $x \in A$.

Functions of several arguments

- Let $F : A \times B \rightharpoonup C$.
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- We use infix notation for most binary functions: $x+y$ for $+(x,y)$.

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- So a bijection has all four I/O properties:
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- If there is a bijection from A to B
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Examples

- In a concert hall filled to capacity
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- The successor-modulo-12 function over $[0..11]$.
- Let $d(x) = 2x$.

$d : \mathbb{R} \rightharpoonup \mathbb{R}$ is a bijection.
$d : \mathbb{N} \rightharpoonup \mathbb{N}$ is not: it is not surjective.
$d : \mathbb{N} \rightharpoonup \textit{Even}$ is a bijection.

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- ***Proof.*** We saw above that the properties univalent, total, injective and surjective are all closed under composition.

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  - Transitive: If $F : A \cong B$ and $G : B \cong C$ then $F\,;G : A \cong C$.

SET SIZE

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  It is a strole of genius for finite sets.

  It is not necessary.

  It hinders generalization of size to infinite sets!

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- The composition of injections is an injection, so:
  Theorem. $\preccurlyeq$ is transitive: If $A \preccurlyeq B \preccurlyeq C$ then $A \preccurlyeq C$.

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- $x \mapsto x/1000$ is an embedding of $(0..1000]$ in $(0..1]$.
- $[1..\infty) \preccurlyeq (0..1]$ by the embedding $x \mapsto 1/x$.

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Over the set $\mathbb{R}$ of real numbers:

- Stretch: For $a, b > 0$ we have $(0\,..\,a) \preccurlyeq (0\,..\,b)$ by the injection $x \mapsto bx/a$.

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- $(1..2) \preccurlyeq [1..2] \preccurlyeq (0..3)$ (by identities)
  $\preccurlyeq (1..2)$ (Stretch)

Equipollence

- Recall that A is $\boxed{\textit{equipollent}}$ with B when there is a bijection $j : A \cong B$.
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- Surprisingly, the converse also holds:

  Cantor-Bernstein-Schröder Theorem. (1896/97)
  If $A \preccurlyeq B$ and $B \preccurlyeq A$ then $A \cong B$.

Using CBS

1. The CBS Theorem is useful in proving set equipollence,
   because mutual embeddings are often easier to find than a bijection.

2. We showed that all real number intervals are embedable in each other.
   So by CBS they are all equipollent to each other.

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     where g is the injection $n \mapsto$ binary numeral for n.

Countable sets

- A set A is $\boxed{\textit{denumerable}}$ if $A \cong \mathbb{N}$.
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- A is $\boxed{\textbf{\textit{countable}}}$ if $A \preccurlyeq \mathbb{N}$.
- So A is countable iff it is either finite or denumerable.

Examples of denumerable sets.

1. The set $\mathbb{Z}$ of integers:

   $\mathbb{Z} \cong \mathbb{N}$ by the bijection

   $x \mapsto$ if $x \geqslant 0$ then $2x$ else $-2x - 1$.

   I.e. $\mathbb{Z}$ is listed as $0, -1, 1, -2, 2, -3, \ldots$.

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2. **Seems like all infinite sets are countable. Are they?**

The size of $\mathcal{P}(A)$

- Not all infinite sets are countable!
- **Cantor's Theorem** (1891)
  For all sets A: $\boxed{\mathcal{P}(A) \not\preceq A}$.
- ***Proof.*** We show that for ever set A and function $g : A \to \mathcal{P}(A)$,
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  – In particular, $\mathcal{P}(\mathbb{N}) \not\cong \mathbb{N}$, that is: $\mathcal{P}(\mathbb{N})$ is not countable!

Comments on Cantor's Theorem

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- Compare:
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- Compare:
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- The set $\mathcal{P}^{fin}(\mathbb{N})$ of ***finite*** subsets of $\mathbb{N}$ is $\preccurlyeq \{0,1\}^*$
  by our familiar embedding, e.g. $\{0,2,3\} \mapsto$ **1011**.

But $\{0,1\}^* \preccurlyeq \mathbb{N}$ so $\mathcal{P}^{fin}(\mathbb{N}) \preccurlyeq \mathbb{N}$ by CBS.

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- $\mathcal{P}(\mathbb{N}) \preccurlyeq (0..1)$:

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- By CBS conclude $\mathbb{R} \cong (0..1) \cong \mathcal{P}(\mathbb{N})$.

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  we construct a bijection $j : A \cong B$.
- For $a \in A$ there is a chain

$$a \xrightarrow{f} b_1 \xrightarrow{g} a_1 \xrightarrow{f} b_2 \xrightarrow{g} a_2 \xrightarrow{f} b_3 \cdots$$

⋆ *Proof of CBS*

- Given injections $f : A \to B$ and $g : B \to A$
  we construct a bijection $j : A \cong B$.
- We might also go backwards:

$$\cdots a_{-2} \xrightarrow{f} b_{-2} \xrightarrow{g} a_{-1} \xrightarrow{f} b_{-1} \xrightarrow{g} a \xrightarrow{f} b_1 \xrightarrow{g} a_1 \xrightarrow{f} b_2 \xrightarrow{g} a_2 \xrightarrow{f} b_3 \cdots$$

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- Similarly, each $b \in B$ starts a chain $b \xrightarrow{g} a_1 \xrightarrow{f} b_1 \xrightarrow{g} a_2 \xrightarrow{f} b_2 \xrightarrow{g} a_3 \cdots$,
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- Every $x \in A \cup B$ is in some chain.
  Repetitions, e.g. $a \xrightarrow{f} b \xrightarrow{g} a \xrightarrow{f} b \quad \cdots$ are harmless.

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- For a chain C let $A_c = A \cap C$, $B_c = B \cap C$, and f_c, g_c be the restrictions of f, g to C.

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- If C starts with $b \in B$ then

  the chain above yields $g_c : B_c \cong A_c$:

  $b \xrightarrow{g} a_1 \qquad b_1 \xrightarrow{g} a_2 \qquad b_2 \xrightarrow{g} a_3 \cdots$

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- The union (over all chains) of these bijections is a bijection from A to B.
  QED.