B501, Fall 2023

Assignment 1: Sets and relations

(Due by EOD W Aug 30)

This assignment contains solved practice problems, numbered in red. The assigned problems and sub-problems are numbered in green.

1. (15%) The operation $\bar{\cap}$ of ***co-intersection*** is defined for subsets of U by $A \bar{\cap} B = U - (A \cap B)$.

i. Define the complement in terms of co-intersection.
Solution. $\overline{A} = A \bar{\cap} A$

(a) Define intersection in terms of co-intersection. (Together with the previous part this shows that all basic set operations can be defined in terms of just one operation: co-intersection!)
Solution. $A \cap B = \overline{A \bar{\cap} B} = (A \bar{\cap} B) \bar{\cap} (A \bar{\cap} B)$ (using (i)).

2. (15%) A set $A \subseteq \mathbb{N}$ is ***co-finite*** if its complement $\mathbb{N} - A$ is finite.

i. Give two examples of a co-finite subset of $\mathbb{N}$.
Solution. $\mathbb{N}$, $\{x \in \mathbb{N} \mid x > 7\}$.

(a) Give two examples of subsets of $\mathbb{N}$ that are neither finite nor co-finite.
Solution. The set of even natural numbers, the set of prime.

(b) Exhibit an infinite sequence $A_1 \supset A_2 \supset A_3 \cdots$ of subests of $\mathbb{N}$, each one a proper subset of its predecessor in the list.

Solution. Take $A_i =_{\text{df}} [i..infty)$.

3. (20%) Prove the following statements. You may use the Separation Principle in your proofs.

(a) There is no "set of all sets."
Solution. If U were the set of all sets, then the Separation Principle would let us define a set $R =_{\text{df}} \{x \in U \mid x \notin x\}$ which leads to the same contradiction underlying Russell's Paradox:

$R \in R$	iff	$R \in U$ and $R \notin R$	(by the dfn of R)
	iff	$R \notin R$	(since $R \in U$ by assumption)

(b) There is no such thing as the "set of all singletons".
Solution. If a set S of all singletons exists, by Separation we'd get the set

$$Q = \{\{x\} \in S \mid \{x\} \notin x\}$$

For every singleton set $\{x\}$ we'd have $\{x\} \in Q$ iff $\{x\} \notin x$. Since Q is a legitimate set, we can take x to be Q and conclue that $\{Q\} \in Q$ iff $\{Q\} \notin Q$, a contradiction.

4. (20%) For each of the following sets S and collections $C \subseteq \mathcal{P}(S)$ determine whether C is a partition of S.

   i. $S = \mathbb{N}$; C consists of three sets: the prime numbers, the composite numbers, the singleton $\{0, 1\}$.

      Solution. This is a partition, as every natural number is in exactly one of the three sets.

   (a) $S =$ the adult population of Indiana. $\mathcal{C}$ consists of the three sets: speakers of English, speakers of Spanish, people who speak neither English nor Spanish.

      Solution. Not a partition: there are people in Indiana who speak both languages.

   (b) $S = \mathbb{R}$. $\mathcal{C}$ consists of the three sets: the rational numbers, the irrational numbers, and the integers. (Search on line for definitions, if you need them.)

      Solution. Not a partition: the integers are rational numbers.

5. (30%) The following problem shows that partitions and equivalence relations are two sides of the same coin.

   Let S be a set.

   (a) Suppose that $\mathcal{C}$ is a partition of S . Define the relation $E \subseteq S \times S$ to hold between $x, y, \in S$ iff they are both in the same $A \in \mathcal{C}$. Show that E is an equivalence relation.

      Solution. E is reflexive, since every $x \in S$ is in the same part as itself. It is symmetric, because if x and y are the same part, then y and x are in the same part (surprise!). And it is transitive: if x and y are in the same part, and y and z are in the same part, then x and z are in the same part.

   (b) Suppose that E is an equivalence relation on S . Show that the collection of equivalence classes of E is a partition of S .

      Solution. The equivalence classes are not empty, by their definition. Every $x \in S$ is in a class, namely $[x]$. And that class is unique, because if $x \in [y]$ then $x \equiv y$ by the definition of y , and so $[x] = [y]$.